
Contents

1	Introduction and mathematical backgrounds	1
1.1	On the book	1
1.1.1	Scope and structure	1
1.1.2	Prerequisites	4
1.1.3	General conventions	4
1.2	On Quantum Mechanics	5
1.2.1	QM as a mathematical theory	5
1.2.2	QM in the panorama of contemporary Physics	7
1.3	Backgrounds on general topology	10
1.3.1	Open/closed sets and basic point-set topology	10
1.3.2	Convergence and continuity	13
1.3.3	Compactness	14
1.3.4	Connectedness	15
1.4	Round-up on measure theory	16
1.4.1	Measure spaces	16
1.4.2	Positive σ -additive measures	19
1.4.3	Integration of measurable functions	22
1.4.4	Riesz's theorem for positive Borel measures	26
1.4.5	Differentiating measures	27
1.4.6	Lebesgue's measure on \mathbb{R}^n	28
1.4.7	The product measure	31
1.4.8	Complex (and signed) measures	32
1.4.9	Exchanging derivatives and integrals	34
2	Normed and Banach spaces, examples and applications	35
2.1	Normed and Banach spaces and algebras	36
2.1.1	Normed spaces and essential properties	36
2.1.2	Banach spaces	40
2.1.3	Example: the Banach space $C(K; \mathbb{K}^n)$, the theorems of Dini and Arzelà-Ascoli	42
2.1.4	Normed algebras, Banach algebras and examples	45

XII Contents

2.2	Operators, spaces of operators, operator norms	53
2.3	The fundamental theorems of Banach spaces	60
2.3.1	The Hahn-Banach theorem and its consequences	60
2.3.2	The Banach-Steinhaus theorem or uniform boundedness principle	64
2.3.3	Weak topologies. * -weak completeness of X'	65
2.3.4	Excursus: the theorem of Krein-Milman, locally convex metrisable spaces and Fréchet spaces	70
2.3.5	Baire's category theorem and its consequences: the open mapping theorem and the inverse operator theorem	73
2.3.6	The closed graph theorem	76
2.4	Projectors	79
2.5	Equivalent norms	80
2.6	The fixed-point theorem and applications	82
2.6.1	The fixed-point theorem of Banach-Caccioppoli	83
2.6.2	Application: local existence and uniqueness for systems of ODEs	87
	Exercises	91
3	Hilbert spaces and bounded operators	97
3.1	Elementary notions, Riesz's theorem and reflexivity	97
3.1.1	Inner product spaces and Hilbert spaces	98
3.1.2	Riesz's theorem and its consequences	102
3.2	Hilbert bases	106
3.3	Hermitian adjoints and applications	119
3.3.1	Hermitian conjugation, or adjunction	119
3.3.2	* -algebras and C^* -algebras	122
3.3.3	Normal, self-adjoint, isometric, unitary and positive operators	127
3.4	Orthogonal projectors and partial isometries	130
3.5	Square roots of positive operators and polar decomposition of bounded operators	135
3.6	The Fourier-Plancherel transform	143
	Exercises	153
4	Families of compact operators on Hilbert spaces and fundamental properties	163
4.1	Compact operators in normed and Banach spaces	164
4.1.1	Compact sets in (infinite-dimensional) normed spaces	164
4.1.2	Compact operators in normed spaces	166
4.2	Compact operators in Hilbert spaces	170
4.2.1	General properties and examples	170
4.2.2	Spectral decomposition of compact operators on Hilbert spaces	172
4.3	Hilbert-Schmidt operators	178
4.3.1	Main properties and examples	179

4.3.2	Integral kernels and Mercer's theorem	187
4.4	Trace-class (or nuclear) operators	190
4.4.1	General properties	190
4.4.2	The notion of trace	193
4.5	Introduction to the Fredholm theory of integral equations	197
	Exercises	204
5	Densely-defined unbounded operators on Hilbert spaces	211
5.1	Unbounded operators with non-maximal domains	211
5.1.1	Unbounded operators with non-maximal domains in normed spaces	212
5.1.2	Closed and closable operators	213
5.1.3	The case of Hilbert spaces: $H \oplus H$ and the operator τ	214
5.1.4	General properties of Hermitian adjoints	215
5.2	Hermitian, symmetric, self-adjoint and essentially self-adjoint operators	217
5.3	Two major applications: the position operator and the momentum operator	221
5.3.1	The position operator	221
5.3.2	The momentum operator	222
5.4	Existence and uniqueness criteria for self-adjoint extensions	227
5.4.1	The Cayley transform and deficiency indices	227
5.4.2	Von Neumann's criterion	231
5.4.3	Nelson's criterion	232
	Exercises	237
6	Phenomenology of quantum systems and Wave Mechanics: an overview	241
6.1	General principles of quantum systems	241
6.2	Particle aspects of electromagnetic waves	243
6.2.1	The photoelectric effect	243
6.2.2	The Compton effect	244
6.3	An overview of Wave Mechanics	246
6.3.1	De Broglie waves	246
6.3.2	Schrödinger's wavefunction and Born's probabilistic interpretation	247
6.4	Heisenberg's uncertainty principle	249
6.5	Compatible and incompatible quantities	250
7	The first 4 axioms of QM: propositions, quantum states and observables	253
7.1	The pillars of the standard interpretation of quantum phenomenology	254
7.2	Classical systems: elementary propositions and states	256
7.2.1	States as probability measures	256
7.2.2	Propositions as sets, states as measures	258

7.2.3	Set-theoretical interpretation of the logical connectives	259
7.2.4	“Infinite” propositions and physical quantities	260
7.2.5	Intermezzo: the theory of lattices	262
7.2.6	The distributive lattice of elementary propositions for classical systems	264
7.3	Quantum propositions as orthogonal projectors	265
7.3.1	The non-distributive lattice of orthogonal projectors on a Hilbert space	266
7.3.2	Recovering the Hilbert space from the lattice	273
7.3.3	Von Neumann algebras and the classification of factors	275
7.4	Propositions and states on quantum systems	275
7.4.1	Axioms A1 and A2 : propositions, states of a quantum system and Gleason’s theorem	276
7.4.2	The Kochen-Specker theorem	283
7.4.3	Pure states, mixed states, transition amplitudes	284
7.4.4	Axiom A3 : post-measurement states and preparation of states	289
7.4.5	Superselection rules and coherent sectors	291
7.4.6	Algebraic characterisation of a state as a noncommutative Riesz theorem	294
7.5	Observables as projector-valued measures on \mathbb{R}	298
7.5.1	Axiom A4 : the notion of observable	298
7.5.2	Self-adjoint operators associated to observables: physical motivation and basic examples	301
7.5.3	Probability measures associated to state/observable couples .	306
	Exercises	308
8	Spectral Theory I: generalities, abstract C^*-algebras and operators in $\mathfrak{B}(\mathcal{H})$	311
8.1	Spectrum, resolvent set and resolvent operator	312
8.1.1	Basic notions in normed spaces	313
8.1.2	The spectrum of special classes of normal operators in Hilbert spaces	317
8.1.3	Abstract C^* -algebras: Gelfand-Mazur theorem, spectral radius, Gelfand’s formula, Gelfand-Najmark theorem	318
8.2	Functional calculus: representations of commutative C^* -algebras of bounded maps	324
8.2.1	Abstract C^* -algebras: continuous functional calculus for self-adjoint elements	324
8.2.2	Key properties of *-homomorphisms of C^* -algebras, spectra and positive elements	328
8.2.3	Commutative Banach algebras and the Gelfand transform .	331
8.2.4	Abstract C^* -algebras: continuous functional calculus for normal elements	336
8.2.5	C^* -algebras of operators in $\mathfrak{B}(\mathcal{H})$: functional calculus for bounded measurable functions	338

8.3	Projector-valued measures (PVMs)	346
8.3.1	Spectral measures, or PVMs	346
8.3.2	Integrating bounded measurable functions in a PVM	349
8.3.3	Properties of integrals of bounded maps in PVMs	355
8.4	Spectral theorem for normal operators in $\mathfrak{B}(\mathcal{H})$	361
8.4.1	Spectral decomposition of normal operators in $\mathfrak{B}(\mathcal{H})$	362
8.4.2	Spectral representation of normal operators in $\mathfrak{B}(\mathcal{H})$	366
8.5	Fuglede's theorem and consequences	374
8.5.1	Fuglede's theorem	374
8.5.2	Consequences	376
	Exercises	377
9	Spectral theory II: unbounded operators on Hilbert spaces	381
9.1	Spectral theorem for unbounded self-adjoint operators	381
9.1.1	Integrating unbounded functions in spectral measures	382
9.1.2	Von Neumann algebra of a bounded normal operator	394
9.1.3	Spectral decomposition of unbounded self-adjoint operators	395
9.1.4	Example with pure point spectrum: the Hamiltonian of the harmonic oscillator	403
9.1.5	Examples with pure continuous spectrum: the operators position and momentum	407
9.1.6	Spectral representation of unbounded self-adjoint operators	408
9.1.7	Joint spectral measures	409
9.2	Exponential of unbounded operators: analytic vectors	411
9.3	Strongly continuous one-parameter unitary groups	415
9.3.1	Strongly continuous one-parameter unitary groups, von Neumann's theorem	416
9.3.2	One-parameter unitary groups generated by self-adjoint operators and Stone's theorem	419
9.3.3	Commuting operators and spectral measures	426
	Exercises	430
10	Spectral Theory III: applications	433
10.1	Abstract differential equations in Hilbert spaces	433
10.1.1	The abstract Schrödinger equation (with source)	435
10.1.2	The abstract Klein-Gordon/d'Alembert equation (with source and dissipative term)	441
10.1.3	The abstract heat equation	449
10.2	Hilbert tensor products	452
10.2.1	Tensor product of Hilbert spaces and spectral properties	452
10.2.2	Tensor product of operators (typically unbounded) and spectral properties	458
10.2.3	An example: the orbital angular momentum	460
10.3	Polar decomposition theorem for unbounded operators	463

10.3.1	Properties of operators A^*A , square roots of unbounded positive self-adjoint operators	464
10.3.2	Polar decomposition theorem for closed and densely-defined operators	468
10.4	The theorems of Kato-Rellich and Kato	470
10.4.1	The Kato-Rellich theorem	470
10.4.2	An example: the operator $-\Delta + V$ and Kato's theorem	472
	Exercises	478
11	Mathematical formulation of non-relativistic Quantum Mechanics	481
11.1	Round-up on axioms A1, A2, A3, A4 and superselection rules	481
11.2	Axiom A5: non-relativistic elementary systems	488
11.2.1	The canonical commutation relations (CCRs)	490
11.2.2	Heisenberg's uncertainty principle as a theorem	491
11.3	Weyl's relations, the theorems of Stone-von Neumann and Mackey	492
11.3.1	Families of operators acting irreducibly and Schur's lemma	493
11.3.2	Weyl's relations from the CCRs	494
11.3.3	The theorems of Stone-von Neumann and Mackey	502
11.3.4	The Weyl * -algebra	504
11.3.5	Proof of the theorems of Stone-von Neumann and Mackey	508
11.3.6	More on "Heisenberg's principle": weakening the assumptions and extension to mixed states	514
11.3.7	The Stone-von Neumann theorem revisited, via the Heisenberg group	516
11.3.8	Dirac's correspondence principle and Weyl's calculus	517
	Exercises	520
12	Introduction to Quantum Symmetries	523
12.1	Definition and characterisation of quantum symmetries	523
12.1.1	Examples	525
12.1.2	Symmetries in presence of superselection rules	526
12.1.3	Kadison symmetries	527
12.1.4	Wigner symmetries	529
12.1.5	The theorems of Wigner and Kadison	531
12.1.6	The dual action of symmetries on observables	540
12.2	Introduction to symmetry groups	546
12.2.1	Projective and projective unitary representations	546
12.2.2	Projective unitary representations are unitary or antiunitary	551
12.2.3	Central extensions and quantum group associated to a symmetry group	552
12.2.4	Topological symmetry groups	554
12.2.5	Strongly continuous projective unitary representations	559
12.2.6	A special case: the topological group \mathbb{R}	562
12.2.7	Round-up on Lie groups and algebras	567

12.2.8 Symmetry Lie groups, theorems of Bargmann, Gårding, Nelson, FS ³	575
12.2.9 The Peter-Weyl theorem	585
12.3 Examples	591
12.3.1 The symmetry group $SO(3)$ and the spin	591
12.3.2 The superselection rule of the angular momentum	595
12.3.3 The Galilean group and its projective unitary representations	596
12.3.4 Bargmann's rule of superselection of the mass	602
Exercises	605
13 Selected advanced topics in Quantum Mechanics	611
13.1 Quantum dynamics and its symmetries.....	612
13.1.1 Axiom A6 : time evolution	612
13.1.2 Dynamical symmetries	614
13.1.3 Schrödinger's equation and stationary states	617
13.1.4 The action of the Galilean group in <i>position representation</i> .	625
13.1.5 Review of scattering processes	627
13.1.6 The evolution operator in absence of time homogeneity and Dyson's series	634
13.1.7 Antiunitary time reversal	637
13.2 The time observable and Pauli's theorem. POVMs in brief	639
13.2.1 Pauli's theorem	640
13.2.2 Generalised observables as POVMs	640
13.3 Dynamical symmetries and constants of motion	643
13.3.1 Heisenberg's picture and constants of motion	643
13.3.2 Detour: Ehrenfest's theorem and related issues.....	647
13.3.3 Constants of motion associated to symmetry Lie groups and the case of the Galilean group	650
13.4 Compound systems and their properties	654
13.4.1 Axiom A7 : compound systems	654
13.4.2 Entangled states and the so-called "EPR paradox"	655
13.4.3 Bell's inequalities and their experimental violation	658
13.4.4 EPR correlations cannot transfer information	661
13.4.5 Decoherence as a manifestation of the macroscopic world ..	663
13.4.6 Axiom A8 : compounds of identical systems	664
13.4.7 Bosons and Fermions	666
Exercises	669
14 Introduction to the Algebraic Formulation of Quantum Theories	671
14.1 Introduction to the algebraic formulation of quantum theories	671
14.1.1 Algebraic formulation and the GNS theorem	672
14.1.2 Pure states and irreducible representations	678
14.1.3 Hilbert space formulation vs algebraic formulation	681
14.1.4 Superselection rules and Fell's theorem	684

XVIII Contents

14.1.5 Proof of the Gelfand-Najmark theorem, universal representations and quasi-equivalent representations	686
14.2 Example of a C^* -algebra of observables: the Weyl C^* -algebra	690
14.2.1 Further properties of Weyl * -algebras	690
14.2.2 The Weyl C^* -algebra	694
14.3 Introduction to Quantum Symmetries within the algebraic formulation.....	695
14.3.1 The algebraic formulation's viewpoint on quantum symmetries	695
14.3.2 Symmetry groups in the algebraic formalism	697
A Order relations and groups.....	701
A.1 Order relations, posets, Zorn's lemma.....	701
A.2 Round-up on group theory	702
B Elements of differential geometry	705
B.1 Smooth manifolds, product manifolds, smooth functions	705
B.2 Tangent and cotangent spaces. Covariant and contravariant vector fields	709
B.3 Differentials, curves and tangent vectors	711
B.4 Pushforward and pullback	712
References	713
Index	719