# The notion of observable and the moment problem for $^*$-algebras and their GNS representations

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**Abstract**. When dealing with unital $^*$-algebras which are not $C^*$-algebras for describing quantum systems, as it happens in QFT, the notion of (quantum) observable has a delicate status. This is because it is generally false that every Hermitian element $a = a^* \in \mathfrak{A}$ is represented by an (essentially) selfadjoint operator $\pi_\omega(a)$ in a given GNS representation induced by an algebraic state $\omega : \mathfrak{A} \to \mathbb{C}$. In general, $\pi_\omega(a)$ results to be simply symmetric admitting many or none selfadjoint extensions. This problem is entangled with another issue concerning the standard physical interpretation of $\omega(a)$ as expectation value of the (abstract) observable $a$ in the state $\omega$. To be effective, that interpretation needs a probability distribution $\mu_\omega^{(a)}$ over $\mathbb{R}$ which may be furnished by the spectral measure of $\overline{\pi_\omega(a)}$ if it is selfadjoint (this spectral measure always exists when $\mathfrak{A}$ is a $C^*$ algebra since $\pi_\omega(a)$ is always selfadjoint if $a = a^*$ in that case). Independently from the existence of a spectral measure, the problem of finding $\mu_\omega^{(a)}$ can be tackled in the framework of the more general Hamburger moment problem, looking for a probability measure whose moments coincide to the known values $\omega(a^n)$ for $n = 0, 1, 2, \ldots$. However, it is possible to prove that, in the general case of a $^*$-algebra which is not $C^*$, there are many such measures for a given pair $(a, \omega)$, independently from the fact that $\pi_\omega(a)$ admits one, many or none selfadjoint extensions. So a discussion on the possible physical meaning of these measures is necessary. This work deals with these issues focusing on the full physical information provided by $a$, $\mathfrak{A}$, and $\omega$, in particular by the perturbed states $\mathfrak{A} \ni c \mapsto \omega_b(c) := \omega(b^*cb)/\omega(b^*b)$, with $b \in \mathfrak{A}$. The class of associated measures $\mu_{\omega_b}^{(a)}$ solution of the corresponding Hamburger moment problem for the moments $\omega_b(a^n)$ is analyzed. As a first result, we establish that if the measures $\mu_{\omega_b}^{(a)}$ are uniquely determined for every $b$ and a fixed pair $(a, \omega)$, then $\overline{\pi_\omega(a)}$ and every $\overline{\pi_{\omega_b}(a)}$ are selfadjoint. The converse statement is however false. As a second, more elaborated, result we prove that for fixed $a^* = a \in \mathfrak{A}$ and $\omega$, when assuming some physically natural coherence constraints on the measures $\mu_{\omega_b}^{(a)}$ varying $b \in \mathfrak{A}$, the admitted families of constrained measures $\{\mu_{\omega_b}^{(a)}\}_{b\in\mathfrak{A}}$ are one-to-one with all possible positive operator-valued measures (POVM) associated to the symmetric operator $\pi_\omega(a)$ through Naimark's decomposition procedure for symmetric operators. $\overline{\pi_\omega(a)}$ is maximally symmetric if and only if there is only one such measure for every fixed $b$. These unique measures are those induced by the unique POVM of $\overline{\pi_\omega(a)}$, which is a Projection Valued Measure if the operator is selfadjoint. The result suggests that a better physical understanding of the GNS representation for $^*$-algebras (not $C^*$) should adopt the more general notion of observable based on POVMs rather than projection-valued measure.

## Contents



## 1 Introduction

This introductory section has the twofold goal of presenting the problems discussed within this work and introducing part of the mathematical machinery used in the rest of the paper. For the general used notation and conventions not directly explained in the text[1], see Section 1.5.

[1] When a new term is introduced and defined, its name appears in **boldface** style.

### 1.1 $^*$-algebras, states, weights and GNS construction

The structure of unital $^*$-algebra $\mathfrak{A}$ (see, e.g. [6]), whose unit element will be henceforth denoted by $\mathbb{I}$, is the most elementary mathematical machinery to describe and handle the set of observables of a quantum system in the algebraic formalism. Algebraic **observables** are here the elements $a \in \mathfrak{A}$ which are **Hermitian** $a = a^*$. This approach is in particular suitable when dealing with the algebra of *quantum fields* where $\mathfrak{A}$ is generated by smeared *quantum field operators* (see, e.g., [5, 6]).

A more rigid version of the $^*$-algebra approach is based on $C^*$-algebras which involve well known topological features arising from a $C^*$-norm (e.g., see [7, 8] for an elementary introduction and [5] for a number of applications in QFT). In this work we stick to the $^*$-algebra case since the issues we go to describe are proper of that case.

As is well known, an (algebraic) **state** over a unital $^*$-algebra $\mathfrak{A}$ is a linear map $\phi : \mathfrak{A} \to \mathbb{C}$ which is **positive** ($\phi(a^*a) \geq 0$ for $a \in \mathfrak{A}$), and **normalized** ($\phi(\mathbb{I}) = 1$).

For $a = a^* \in \mathfrak{A}$, the physical interpretation of $\phi(a)$ is the *expectation value* of the observable $a$ in the state $\phi$. Before addressing the discussion about the interpretation of $a$ as an observable and of $\phi(a)$ as an expectation value and examining the interplay of these two popular physical assumptions, it is necessary to list few fundamental technical notions and results.

**Definition 1.1** If $\mathfrak{A}$ is a $^*$-algebra with unit $\mathbb{I}$, a **finite weight** on $\mathfrak{A}$ is a linear map $\omega : \mathfrak{A} \to \mathbb{C}$ which is positive and $\omega(\mathbb{I}) \neq 0$. ∎

Evidently $\omega(\mathbb{I}) = \omega(\mathbb{I}\mathbb{I}) = \omega(\mathbb{I}^*\mathbb{I}) > 0$. A finite weight $\omega$ is therefore a non-normalized state and it defines a unique associate state $\widehat{\omega}(a) := \omega(\mathbb{I})^{-1}\omega(a)$ for $a \in \mathfrak{A}$.

The basic link between the algebraic formalism and the Hilbert space formulation of quantum theories where (some) states are represented by vectors in a suitable Hilbert space, observables are represented by (some) selfadjoint operators and expectation values are computed in terms of the scalar product of that space, is provided by a celebrated construction developed by Gelfand, Naimark, and Segal and known as *GNS construction* (see, e.g., [6, 8]). It is valid for every state over a unital $^*$-algebra and trivially extends to finite weights. The construction admits a more sophisticated topological version for $C^*$-algebras as commented at the beginning of the next section.

**Theorem 1.2** **[GNS construction]** *Let $\mathfrak{A}$ be a unital $^*$-algebra and $\omega : \mathfrak{A} \to \mathbb{C}$ a finite weight. There exists a quadruple $(\mathsf{H}_\omega, \mathcal{D}_\omega, \pi_\omega, \psi_\omega)$ called* **GNS quadruple** *of $(\mathfrak{A}, \omega)$, where*

*(1) $\mathsf{H}_\omega$ is a Hilbert space whose scalar product is denoted by $\langle\,|\,\rangle$,*

*(2) $\mathcal{D}_\omega \subset \mathsf{H}_\omega$ is a dense subspace,*

*(3) $\pi_\omega : \mathfrak{A} \ni a \mapsto \pi_\omega(a)$ – with $\pi_\omega(a) : \mathcal{D}_\omega \to \mathcal{D}_\omega$ – is a unital algebra representation with the property that $\pi_\omega(a^*) \subset \pi_\omega(a)^\dagger$, where $^\dagger$ denotes the adjoint in $\mathsf{H}_\omega$,*

*(4) $\psi_\omega \in \mathsf{H}_\omega$ is a vector such that*

*(i)* $\mathcal{D}_\omega = \pi_\omega(\mathfrak{A})\psi_\omega$,

*(ii)* $\omega(\pi_\omega(a)) = \langle\psi_\omega|\pi_\omega(a)\psi_\omega\rangle$ *for every* $a \in \mathfrak{A}$*, in particular* $||\psi_\omega||^2 = \omega(\mathbb{I})$.

*If* $(\mathsf{H}, \mathcal{D}, \pi, \psi)$ *satisfies (1)-(4), then there is a surjective isometric map* $U : \mathsf{H}_\omega \to \mathsf{H}$ *such that* $U(\mathcal{D}_\omega) = \mathcal{D}$*,* $\pi(a) = U\pi_\omega(a)U^{-1}$ *for* $a \in \mathfrak{A}$*, and* $\psi = U\psi_\omega$*.*

If $\mathfrak{A}$ is a unital $^*$-algebra, the set

$$G_{(\mathfrak{A},\omega)} := \{a \in \mathfrak{A} \mid \omega(a^*a) = 0\} \tag{1}$$

is a **left-ideal** of $\mathfrak{A}$ (a linear subspace such that $ba \in G_{(\mathfrak{A},\omega)}$ if $a \in G_{(\mathfrak{A},\omega)}$ and $b \in \mathfrak{A}$) as elementary consequence of Cauchy-Schwartz inequality and positivity of $\omega$. $G_{(\mathfrak{A},\omega)}$ is called **Gelfand ideal**. It will result useful in this work observing that the proof of the GNS theorem (e.g. see [8]) leads to

$$\mathfrak{A}/G_{(\mathfrak{A},\omega)} = \mathcal{D}_\omega\,, \quad \pi_\omega(a)[b] = [ab] \quad \text{if } a \in \mathfrak{A}\,, [b] \in \mathfrak{A}/G_{(\mathfrak{A},\omega)}\,, \quad \text{and } \psi_a = [\mathbb{I}]. \tag{2}$$

In particular, $\ker(\pi_\omega) \subset G_{(\mathfrak{A},\omega)}$ and $\pi_\omega$ is faithful when $G_{(\mathfrak{A},\omega)} = \{0\}$[2].

### *1.2 Issue A: interpretation of $\pi_\omega(a)$ as observable*

When $\mathfrak{A}$ is a unital $C^*$-algebra (see, e.g. [7]), $\pi_\omega(a)$ continuously extends to a $^*$-algebra representation of $\mathfrak{A}$ to $\mathfrak{B}(\mathsf{H}_\omega)$. The extended representation denoted by the same symbol $\pi_\omega$ satisfies $||\pi_\omega(a)|| \leq ||a||$ if $a \in \mathfrak{A}$, where $=$ is valid for all $a$ if and only if $\pi_\omega$ is injective. In particular, for $C^*$-algebras, it holds $\pi_\omega(a^*) = \pi_\omega(a)^\dagger$ if $a \in \mathfrak{A}$. Therefore, Hermitian elements of $\mathfrak{A}$ are truly represented by (bounded) selfadjoint operators as it happens in the standard Hilbert space formulation of quantum theories. Here, it seems very appropriate to think of a Hermitian element $a \in \mathfrak{A}$ as an (abstract or algebraic) *observable*.

Referring to the more elementary structure of unital $^*$-algebra, the picture becomes more complex. Every operator $\pi_\omega(a)$ has the common dense invariant domain $\mathcal{D}_\omega$ by definition and it is closable, since its adjoint operator $\pi_\omega(a)^\dagger$ extends $\pi_\omega(a^*)$ which has again the dense domain $\mathcal{D}_\omega$. As a consequence, $\pi_\omega(a)$ is a *symmetric operator* provided that $a = a^*$. If $\pi_\omega(a)$ is *essentially selfadjoint* for *every* $\omega$, then we are again authorized to think of $a$ as an (abstract or algebraic) observable.

A very strong sufficient condition (by no means necessary!) assuring essential selfadjointeness of $\pi_\omega(a)$ for a fixed Hermitian element $a \in \mathfrak{A}$ and *every* weight $\omega$ is that [11] there exist $b_\pm \in \mathfrak{A}$ such that $(a \pm i\mathbb{I})b_\pm = \mathbb{I}$ (equivalently $b'_\pm(a \pm i\mathbb{I}) = \mathbb{I}$, where $b'_\pm = b^*_\mp$). This is because the written condition trivially implies that $\mathrm{Ran}(\pi_\omega(a) \pm iI) \supset \mathcal{D}_\omega$ and thus $\mathrm{Ran}(\pi_\omega(a) \pm iI)$ is dense, so that the symmetric operator $\pi_\omega(a)$ is essentially selfadjoint (see, e.g. [7, Thm 5.18]).

However, if $a^* = a \in \mathfrak{A}$, and $\omega$ is a generic state (or weight), we expect that $\pi_\omega(a)$ may admit *different* selfadjoint extensions. Or, worse, that $\pi_\omega(a)$ admits *no* selfadjoint

[2]The converse does not hold, since $\ker(\pi_\omega)$ is a two-sided $^*$-ideal and thus $\ker(\pi_\omega) \subsetneq G_{(\mathfrak{A},\omega)}$ in the general case.

extensions at all (its deficency indices are different). In these situations, the interpretation of $\pi_\omega(a)$ as an observable, and, correspondingly, $a = a^*$ as an abstract observable seems to be disputable. Examples of the two cases, many selfadjoint extensions or none, can be constructed just exploiting standard results of Quantum Mechanics as we go to prove.

**Example 1.3**
**(1)** Let us define the space $\mathscr{S}$ of complex-valued smooth functions with domain $[0,1]$ which vanish at 0 and 1 with all of their derivatives. Consider the unital $^*$-algebra $\mathfrak{A}$ of differential operators acting on the function of the invariant space $\mathscr{S}$, made of all finite linear combinations of finite compositions in arbitrary order of (i) the operator $P := -i\frac{d}{dx}$, (ii) multiplicative operators $f\cdot$ induced by functions $f \in \mathscr{S}$, and (iii) the constantly 1 function again acting multiplicatively and also defining the unit of the algebra. The involution is $A^* := A^\dagger\restriction_{\mathscr{S}}$ ($^\dagger$ being the adjoint in $L^2([0,1],dx) \supset \mathscr{S}$) so that $P^* = P$. We stress that we are here considering $\mathfrak{A}$ as an abstract algebra (i.e., up to isomorphisms of unital $^*$-algebras) independently from the concrete realization we described above. Now consider the state $\omega : \mathfrak{A} \to \mathbb{C}$ defined as, where $dx$ is the Lebesgue measure on $\mathbb{R}$,

$$\omega(A) := \int_0^1 \psi(x)(A\psi)(x)dx \quad A \in \mathfrak{A}\,.$$

Above, $\psi \in \mathscr{S}$ is a fixed non-negative function vanishing *only* at 0 and 1. The GNS structure is easy to be constructed taking advantage of the uniqueness part of GNS theorem,

$$\mathsf{H}_\omega = L^2([0,1],dx)\,, \quad \pi_\omega(A) = A\restriction_{\mathcal{D}_\omega}\,, \quad \psi_\omega := \psi\,,$$

and $\mathcal{D}_\omega$ is a suitable subspace of $\mathscr{S}$ which however includes $C_c^\infty(0,1)$, the space of smooth complex maps $f : [0,1] \to \mathbb{C}$ supported in $(0,1)$, so that $\mathcal{D}_\omega$ is dense in $L^2([0,1],dx)$ as is due. The defect spaces $N_\pm$ of $\pi_\omega(P) = P\restriction_{\mathcal{D}_\omega}$ are

$$N_\pm := \left\{ g_\pm \in L^2([0,1],dx) \;\middle|\; \int_0^1 \overline{g_\pm}(f' \pm f)dx = 0 \quad \forall f \in \mathcal{D}_\omega \right\}$$
$$= \left\{ g_\pm \in L^2([0,1],dx) \;\middle|\; \int_0^1 \overline{g_\pm(x)e^{\mp x}}\left(f(x)e^{\pm x}\right)' dx = 0 \quad \forall f \in \mathcal{D}_\omega \right\} \tag{3}$$

If $\mathcal{D}_\omega$ were replaced by $C_c^\infty(0,1)$ in (3), [7, Lemma 5.30] would imply $g_\pm(x) = ce^{\pm x}$ for $c \in \mathbb{C}$. However these function would also satisfy (3) if $f \in \mathscr{S}$, as one immediately proves per direct inspection. Since $C_c^\infty(0,1) \subset \mathcal{D}_\omega \subset \mathscr{S}$, we conclude that $N_\pm = \text{span}\{e^{\pm x}\}$. Therefore the symmetric operator $\pi_\omega(P)$ is *not* essentially selfadjoint on its GNS domain $\mathcal{D}_\omega$, but it admits a one-parameter class of different selfadjoint extensions according to *von Neumann's extension theorem* (see, e.g. [7, Thm 5.37]).
**(2)** Let us define the space $\mathscr{E}$ of complex-valued smooth functions with domain $[0,+\infty)$ which vanish at 0 with all of their derivatives and tend to 0 with all of their derivatives for $x \to +\infty$ faster than every negative power of $x$. Consider the unital $^*$-algebra $\mathfrak{B}$ of differential operators acting on the function of the invariant space $\mathscr{E}$, made of all finite

linear combinations of finite compositions in arbitrary order of (i) the operator $P := -i\frac{d}{dx}$, (ii) multiplicative operators $f\cdot$ induced by functions $f \in \mathscr{E}$, and (iii) the constantly 1 function again acting multiplicatively and also defining the unit of the algebra. The involution is $A^* := A^\dagger\restriction_{\mathscr{E}}$ ($^\dagger$ being the adjoint in $L^2([0,+\infty), dx) \supset \mathscr{E}$) so that $P^* = P$. As before, we are here considering $\mathfrak{B}$ as an abstract algebra (i.e., up to isomorphisms of unital $^*$-algebras) independently from the concrete realization we presented above. Next consider the state $\phi : \mathfrak{B} \to \mathbb{C}$ defined as

$$\phi(A) := \int_0^{+\infty} \chi(x)(A\chi)(x)dx \quad A \in \mathfrak{B}\,.$$

Above, $\chi \in \mathscr{E}$ is a fixed non-negative function vanishing *only* at 0. Uniqueness part of the GNS theorem proves that the GNS structure is

$$\mathsf{H}_\phi = L^2([0,+\infty), dx)\,, \quad \pi_\phi(A) = A\restriction_{\mathcal{D}_\phi}\,, \quad \psi_\phi := \chi\,,$$

and $\mathcal{D}_\phi$ is a suitable subspace of $\mathscr{E}$ which however includes $C_c^\infty(0,+\infty)$, the space of smooth complex maps $f : [0,+\infty) \to \mathbb{C}$ whose supports are included in $(0,+\infty)$. Hence $\mathcal{D}_\phi$ is dense in $L^2([0,+\infty), dx)$ as is due. The defect spaces $N_\pm$ of $\pi_\phi(P) = P\restriction_{\mathcal{D}_\phi}$ can be computed easily

$$N_\pm := \left\{ g_\pm \in L^2([0,+\infty), dx) \;\middle|\; \int_0^{+\infty} \overline{g_\pm}(f' \pm f)dx = 0 \quad \forall f \in \mathcal{D}_\phi \right\}$$
$$= \left\{ g_\pm \in L^2([0,+\infty), dx) \;\middle|\; \int_0^{+\infty} \overline{g_\pm(x)e^{\mp x}}\left(f(x)e^{\pm x}\right)' dx = 0 \quad \forall f \in \mathcal{D}_\phi \right\} \qquad (4)$$

If $\mathcal{D}_\phi$ were replaced by $C_c^\infty(0,+\infty)$ in (4), [7, Lemma 5.30] would imply $g_\pm(x) = ce^{\pm x}$ for $c \in \mathbb{C}$. Evidently $ce^x$ cannot accepted as an element of $L^2([0,+\infty), dx)$ unless $c = 0$, so that, since $C_c^\infty(0,+\infty) \subset \mathcal{D}_\phi$, we conclude that $N_+ = \{0\}$. The functions $ce^{-x}$ would fulfill (4) even if $\mathcal{D}_\phi$ were replaced by $\mathscr{E}$, as one immediately proves per direct inspection. Since $C_c^\infty(0,+\infty) \subset \mathcal{D}_\phi \subset \mathscr{E}$, we conclude that $N_- = \text{span}\{e^{-x}\}$ whereas $N_+ = \{0\}$. Therefore the symmetric operator $\pi_\phi(P)$ is *not* essentially selfadjoint on its GNS domain $\mathcal{D}_\phi$ and it does *not* admit selfadjoint extensions. More strongly, as $N_- = \{0\}$ but $N_+ \neq \{0\}$, we have that $\overline{\pi_\phi(P)}$ is *maximally symmetric*: it does not admit proper symmetric extensions [1, Thm 3, p.97]. ■

In summary, it seems that, dealing with $^*$-algebras which are not $C^*$, there is not a perfect match between the algebraic notion of observable (Hermitian element of $\mathfrak{A}$) and that in the (GNS) Hilbert space formulation (selfadjoint operator): Hermitian elements of $^*$-algebras are usually represented by merely symmetric operators in the GNS representations with many or none selfadjoint extensions. To authors' knowledge, this physically crucial problem has not been investigated as it should deserve in the literature.

### *1.3 Issue B: interpretation of $\omega(a)$ as expectation value and the moment problem*

Let us pass to discuss the interpretation of $\omega(a)$ as *expectation value* for $a = a^* \in \mathfrak{A}$, where $\omega$ is a state or more generally a *finite weight* $\omega$[3]. To rigorously accept this folk physical interpretation, we should assume that the pair $(a, \omega)$ admits a physically meaningful uniquely associated positive $\sigma$-additive measure $\mu_\omega^{(a)}$ over $\mathbb{R}$ such that

$$\omega(a) = \int_{\mathbb{R}} \lambda d\mu_\omega^{(a)}(\lambda) \,. \tag{5}$$

It is natural to also suppose that $\mu$ is defined on the Borel $\sigma$-algebra $\mathscr{B}(\mathbb{R})$, since this is the case for measures arising from the spectral theory as it the standard in quantum theories. Identity (5) is far from being able to determine $\mu_\omega^{(a)}$. However, the structure of $^*$-algebra permits us to define real polynomials of observables and $\omega$ does assign values to all those observables. So we are authorized to assume that the values $\omega(a^n) \in \mathbb{R}$ are known for every $n \in \mathbb{N} := \{0, 1, 2, \ldots\}$ and we can reinforce the constraint (5) to

$$\omega(a^n) = \int_{\mathbb{R}} \lambda^n d\mu_\omega^{(a)}(\lambda) \quad \text{for every } n \in \mathbb{N} \,, \tag{6}$$

since $a^n$ has the natural interpretation of the observable whose values are $\lambda^n$ if $\lambda$ is a value attained by $a$. This way, $\omega(a^n)$ is interpreted as the $n$-th *moment* of the unknown measure $\mu_\omega^{(a)}$. Finding a finite *positive Radon measure* over $\mathbb{R}$ when its moments are fixed is a quite famous problem named *Hamburger moment problem*, very treated in the pure mathematical literature (see [13] for a modern textbook on the subject).

**Remark 1.4** A **positive Radon measure** is a positive $\sigma$-additive measure defined on the Borel sets of a Hausdorff locally-compact space (here $\mathbb{R}$ equipped with the Euclidean topology) which is both outer and inner regular and assigns a finite value to every compact set. All measures considered above are necessarily finite because $\omega(a^0) = \omega(\mathbb{I})$ exists in $[0, +\infty)$ by hypothesis. In $\mathbb{R}^n$, all finite positive $\sigma$-additive Borel measures are automatically Radon in view of [15, Thm 2.18]. Therefore, "positive Radon measure" can be equivalently replaced by "positive $\sigma$-additive Borel measure" in the rest of the discussion related to the moment problem. ■

At this juncture, for a given Hermitian $a \in \mathfrak{A}$ and a given finite weight $\omega : \mathfrak{A} \to \mathbb{C}$ , we should tackle two problems if we want to insist with the standard interpretation of $\omega(a)$ as expectation value.

(M1) Does a positive $\sigma$-additive Borel measure $\mu_\omega^{(a)}$ over $\mathbb{R}$ satisfying (6) exist?

(M2) Is it unique?

[3]In this case, the meaning of expectation value would be actually reserved to $\omega(\mathbb{I})^{-1}\omega(a)$, though, for shortness, we improperly also call $\omega(a)$ expectation value in the rest of the work.

Issues A and B are interrelated into several ways. Here is a first example of that interplay arising when facing (M1) and (M2). If $\pi_\omega(a)$ is essentially selfadjoint, a measure as in (M1) directly arises form the GNS construction. It is simply constructed out of the PVM $P^{(\overline{\pi_\omega(a)})} : \mathscr{B}(\mathbb{R}) \to \mathfrak{B}(\mathsf{H}_\omega)$ of the selfadjoint operator $\overline{\pi_\omega(a)}$ over $\mathsf{H}_\omega$:

$$\mu_\omega^{(a)}(E) := \langle \psi_\omega | P^{(\overline{\pi_\omega(a)})}(E)\psi_\omega \rangle \,, \quad E \in \mathscr{B}(\mathbb{R}) \,. \tag{7}$$

This opportunity is always present if $\mathfrak{A}$ is a unital $C^*$-algebra, since $\pi_\omega(a) \in \mathfrak{B}(\mathsf{H}_\omega)$ is selfadjoint in that case. Concerning (M2), it is possible to prove that the measure defined in (7) is also *unique* when $\mathfrak{A}$ is $C^*$.

**Proposition 1.5** *Let $\mathfrak{A}$ be a unital $^*$-algebra, $a = a^* \in \mathfrak{A}$, and $\omega : \mathfrak{A} \to \mathbb{C}$ is a finite weight. The following facts hold.*

- **(a)** *If $\pi_\omega(a)$ is essentially selfadjoint, then there exists a (necessarily finite) positive $\sigma$-additive Borel measure $\mu_\omega^{(a)} : \mathscr{B}(\mathbb{R}) \to [0, +\infty)$ satisfying (6).*
- **(b)** *If furthermore $\mathfrak{A}$ is a $C^*$ algebra, then the found measure is the unique positive $\sigma$-additive Borel measure on $\mathbb{R}$ satisfying (6).*

*Proof.* (a) The measure (7) is a finite positive $\sigma$-additive Borel measure over $\mathbb{R}$ due to standard properties of spectral measures (see. e.g. [7]), it also satisfies (6). Indeed, since $D(\overline{\pi_\omega(a)}^n) \supset D(\pi_\omega(a)^n) \supset D(\pi_\omega(a^n)) = \mathcal{D}_\omega \ni \psi_\omega$, from spectral theory (see. e.g. [7]) we have

$$\int_{\mathbb{R}} \lambda^n d\mu_\omega^{(a)} = \langle \psi_\omega | \overline{\pi_\omega(a)}^n \psi_\omega \rangle = \langle \psi_\omega | \pi_\omega(a^n) \psi_\omega \rangle = \omega(a^n) \,.$$

(b) Let us suppose that $\mathfrak{A}$ is also a $C^*$-algebra. Since $|\omega(a^n)| \leq \omega(\mathbb{I})\|a\|^n$, *Carleman's condition* [13, Corollary 4.10] assures that there exists at most one positive Radon measure satisfying (6). Observe that $\mu_\omega^{(a)}$ is a positive Radon measure in view of Remark 1.4. □

There are cases of unital $^*$-algebras $\mathfrak{A}$ and states (or weights) $\omega : \mathfrak{A} \to \mathbb{C}$ such that Hermitian elements $a \in \mathfrak{A}$ exist whose associated GNS operator $\pi_\omega(a)$ is not essentially selfadjoint (see Example 1.3 above). In this situation Proposition 1.5 cannot be directly exploited. If $\pi_\omega(a)$ admits selfadjoint extensions (it is sufficient that it commutes with a *conjugation*) each of these selfadjoint extensions induces a measure $\mu_\omega^{(a)}$ as above. However, measures satisfying (M1) for the pair $(a, \omega)$ do exist, and they are not necessarily unique, even if $\pi_\omega(a)$ does *not* admit any selfadjoint extension ((2) Example 1.3 above) making the situation even more intricate.

**Proposition 1.6** *Let $\mathfrak{A}$ be a unital $^*$-algebra, $a = a^* \in \mathfrak{A}$, and $\omega : \mathfrak{A} \to \mathbb{C}$ a finite weight. Then there exists a positive $\sigma$-additive Borel measure $\mu_\omega^{(a)} : \mathscr{B}(\mathbb{R}) \to [0, +\infty)$ satisfying (6).*

*Proof.* A general proof is based on the classical solution of the existence part of Hamburger moment problem [14, Thm X.4], simply noticing that positivity of $\omega$ and its linearity implies that the set of candidate moments $m_n := \omega(a^n)$ satisfies the hypotheses of the theorem. However, we intend to provide here a direct construction (which is nothing but the proof of the quoted theorem written with a different language). Define a subspace $\mathcal{D}_\omega^{(a)}$ of $\mathcal{D}_\omega$ as follows $\mathcal{D}_\omega^{(a)} := \{\pi_\omega(p(a))\psi_\omega \mid p : \mathbb{R} \to \mathbb{C}$ polynomial of arbitrary finite degree$\}$ and define the closed subspace $\mathsf{H}_\omega^{(a)}$ of $\mathsf{H}_\omega$ as the closure of $\mathcal{D}_\omega^{(a)}$. By construction, $\pi_\omega(a)$ leaves $\mathcal{D}_\omega^{(a)}$ invariant and $\pi_\omega(a)\restriction_{\mathcal{D}_\omega^{(a)}}$ is symmetric in $\mathsf{H}_\omega^{(a)}$. Finally, $\pi_\omega \restriction_{\mathcal{D}_\omega^{(a)}}$ commutes with the *conjugation* $C : \mathsf{H}_\omega^{(a)} \to \mathsf{H}_\omega^{(a)}$ obtained as the unique continuous extension of the antilinear isometric involutive map $\pi_\omega(p(a))\psi_\omega \mapsto \pi_\omega(p(a))^\dagger\psi_\omega$ over $\mathcal{D}_\omega^{(a)}$ (use the fact that $a = a^*$). In view of the von Neumann criterion (see, e.g. [7, Thm 5.43]), $\pi_\omega \restriction_{\mathcal{D}_\omega^{(a)}}$ admits selfadjoint extensions $\widehat{a}_\omega$ on $\mathsf{H}_\omega^{(a)}$. The same argument exploited to prove (a) in Proposition 1.5 restricted to $\mathsf{H}_\omega^{(a)}$ concludes the proof because

$$\mu_\omega^{(a)}(E) := \langle\psi_\omega|P^{(\widehat{a}_\omega)}(E)\psi_\omega\rangle\,, \quad E \in \mathscr{B}(\mathbb{R}) \tag{8}$$

satisfies all requirements for every such selfadjoint extension $\widehat{a}_\omega$. □

In turn, the proof of the Proposition 1.6 raises another issue. Are all measures $\mu_\omega^{(a)}$ associated with a pair $(a, \omega)$ spectrally constructed from selfadjoint extensions over $\mathsf{H}_\omega^{(a)}$ of $\pi_\omega(a)\restriction_{\mathcal{D}_\omega^{(a)}}$ when this operator admits such extensions? The answer is once again negative as it can be grasped from the detailed discussion about the moment problem in the operatorial approach appearing in Ch.6 of [13]. All measures satisfying (M1) are in fact spectrally obtained by *enlarging* the Hilbert space $\mathsf{H}_\omega^{(a)}$ *without reference to the original common GNS Hilbert space* $\mathsf{H}_\omega$.

This fact eventually suggests that, in principle, there could be a plethora of measures associated with $(a, \omega)$ as solutions of the moment problem with dubious physical meaning, because they are vaguely related with the underpinning physical theory described by $\mathfrak{A}$ and $\omega$. Our feeling is that focusing on the *whole* class of the measures satisfying (M1) for a given pair $(a, \omega)$ is probably a wrong approach to tackle the problem of the interpretation of $\omega(a)$ as expectation value. Further physical meaningful information has to be added in order to reduce the number of elements of the family of measures.

### *1.4 Structure of this work*

This paper is organized as follows. After having fixed some notation and conventions, Sect.2 will introduce the notion of perturbed weight which will play a crucial role in the rest of the analysis developed in the paper. In particular, that notion will be exploited to prove a first result concerning essential selfadjointness of $\pi_\omega(a)$ when the measures solving the moment problem for every perturbed weight are unique. Sect.3 contains a recap of the basic theory of POVMs and the theory of generalized selfadjoint extensions of symmetric operators (some complements appears also in the appendix A.) Sect. 4 is the core of the work where are established the main theorems arising from the two issues discussed in the introduction. Here the notions of perturbed states and the mathematical technology of

POVMs meet proving that a strong interplay exists between the moment problem for the deformed weights and the notion of extended observable in terms of POVMs. Sect.5 offers a summary of the results established in the paper and present some open issue. Appendix A includes some complements about reducing subspaces, generalized selfadjoint extensions of symmetric operators and offers the proofs of some technical propositions.

### *1.5 Notation and conventions*

We adopt standard notation and definitions and, barring the symbol of the adjoint operator and that of scalar product, they are the same as in [7]. In particular, an operator in a Hilbert space $\mathsf{H}$ is indicated by $A : D(A) \to \mathsf{H}$, or simply $A$, where the domain $D(A)$ is always supposed to be a linear subspace of $\mathsf{H}$. The scalar product $\langle x|y\rangle$ of a Hilbert space is supposed to be antilinear in the *left* entry. $\mathfrak{B}(\mathsf{H})$ denotes the $C^*$-algebra of bounded operators $A$ in Hilbert space $\mathsf{H}$ with $D(A) = \mathsf{H}$. $\mathscr{L}(\mathsf{H})$ indicates the lattice of orthogonal projectors over the Hilbert space $\mathsf{H}$. The adjoint of an operator $A$ in a Hilbert space is always denoted by $A^\dagger$, while the symbol $a^*$ indicates the adjoint of an element $a$ of a $^*$-algebra. A representation of unital $^*$-algebras is supposed to preserve the identity.
The closure of a closable operator $A : D(A) \to \mathsf{H}$ is indicated by $\overline{A}$. The symbol $A \subset B$ permits the case $A = B$. If $A$ and $B$ are operators $A \subset B$ means that $D(B) \supset D(A)$ and $B\restriction_{D(A)}= A$. An operator $A$ in a Hilbert space $\mathsf{H}$ is said to be **Hermitian** if $\langle Ax|y\rangle = \langle x|Ay\rangle$ for every $x, y \in D(A)$. A Hermitian operator $A$ is **symmetric** if $D(A)$ is dense in $\mathsf{H}$ (equivalently, $A \subset A^\dagger$). A symmetric operator is **selfadjoint** if $A = A^\dagger$, **essentially selfadjoint** if it admits a unique selfadjoint extension (equivalently, if $\overline{A}$ is selfadjoint), in this case $\overline{A}$ is the unique selfadjoint extension of $A$. A **conjugation** in the Hilbert space $\mathsf{H}$ is an *antilinear* isometric map $C : \mathsf{H} \to \mathsf{H}$ such that $CC = I$, where $I$ always denotes the **identity operator** $I : \mathsf{H} \ni x \mapsto x \in \mathsf{H}$.

## 2 Perturbations of states/weights

In the discussion developed in the introduction, when presenting the issues A and B, we completely overlooked the physically meaningful fact that other elements $b \in \mathfrak{A}$ than $a$ exist. These elements can be used to generate new weights $\omega_b$ out of $\omega$ viewed as perturbation of it: $\omega_b(a) := \omega(b^*ab)$. When we are given the triple $\mathfrak{A}, a, \omega$ (with $a = a^*$) we also know the formal expectation values $\omega_b(a)$. We expect that these perturbed states and the associated measures $\mu^{(a)}_{\omega_b}$ solving the moment problem with respect to $\omega_b$ should enter the game. Theorem 2.3 below shows that it is the case.

### *2.1 Perturbations of a finite weight*

**Definition 2.1** If $\omega$ is a finite weight (or a state) over $\mathfrak{A}$, we will denote by $\omega_b$ the finite weight, called $b$**-perturbation** of $\omega$

$$\omega_b(a) := \omega(b^*ab) \quad \forall a \in \mathfrak{A}\,, \tag{9}$$

where $b \in \mathfrak{A}$. The limit case of the zero functional $\omega_b$ obtained from $b$ with $\omega(b^*b) = 0$ is included, and we call that $\omega_b$ **singular perturbation**. ■

From the final uniqueness part of Theorem 1.2, the GNS structure of a non-singular perturbation $\omega_b$ is evidently

$$(\mathsf{H}_{\omega_b}, \mathcal{D}_{\omega_b}, \pi_{\omega_b}, \psi_{\omega_b}) = \left(\overline{\pi_\omega(\mathfrak{A})\psi_{\omega_b}}, \pi_\omega(\mathfrak{A})\psi_{\omega_b}, \pi_\omega\restriction_{\mathcal{D}_{\omega_b}}, \pi_\omega(b)\psi_\omega\right) . \tag{10}$$

If $\omega_b$ is singular we *define*,

$$(\mathsf{H}_{\omega_b}, \mathcal{D}_{\omega_b}, \pi_{\omega_b}, \psi_{\omega_b}) := (\{0\}, \{0\}, 0, 0) . \tag{11}$$

**Remark 2.2** Observe that $\omega_b = \omega_{b'}$ if $[b] = [b']$ referring to the Gelfand-ideal quotient. Therefore the weights $\omega_b$ would be better labelled by the vectors in $\mathcal{D}_\omega = \mathfrak{A}/G_{(\mathfrak{A},\omega)}$. ■

### *2.2 Selfadjointness of $\overline{\pi_\omega(a)}$ and uniqueness of moment problems for perturbed weights*

The fact that focusing to the perturbations $\omega_b$ goes towards the correct direction in order to clarify issues A and B is evident from the following result, the first main result of the paper, which connects uniqueness of the measures $\mu^{(a)}_{\omega_b}$ with selfadjointness of $\overline{\pi_\omega(a)}$.

**Theorem 2.3** *Let $\mathfrak{A}$ be a unital $^*$-algebra, $a^* = a \in \mathfrak{A}$, and $\omega : \mathfrak{A} \to \mathbb{C}$ a finite weight. Assume that the finite positive $\sigma$-additive Borel measure $\mu^{(a)}_{\omega_b} : \mathscr{B}(\mathbb{R}) \to [0, +\infty)$ solving the moment problem for every non-singular perturbation $\omega_b$*

$$\omega_b(a^n) = \int_{\mathbb{R}} \lambda^n d\mu^{(a)}_{\omega_b}(\lambda) \quad \textit{for every } n \in \mathbb{N}. \tag{12}$$

*is unique*[4]. *The following facts hold for every perturbation $\omega_b$.*

- **(a)** *$\pi_\omega(a)$ is essentially selfadjoint in $\mathsf{H}_\omega$ and, more generally, all perturbed operators $\pi_{\omega_b}(a)$ are essentially selfadjoint in the respective $\mathsf{H}_{\omega_b}$.*
- **(b)** *All measures $\mu^{(a)}_{\omega_b}$ are mastered by the single PVM $P^{(\overline{\pi_\omega(a)})}$ since*

$$\mu^{(a)}_{\omega_b}(E) = \langle \psi_{\omega_b} | P^{(\overline{\pi_\omega(a)})}(E) \psi_{\omega_b} \rangle , \quad E \in \mathscr{B}(\mathbb{R}) , \tag{13}$$

$$P^{(\overline{\pi_{\omega_b}(a)})}(E) = P^{(\overline{\pi_\omega(a)})}(E)\restriction_{\mathsf{H}_{\omega_b}} , \quad E \in \mathscr{B}(\mathbb{R}) . \tag{14}$$

*Proof.* (a) *Carleman's condition* [13, Corollary 4.10] assures that, if $\omega_b$ is singular, only the zero measure $\mu^{(a)}_{\omega_b} = 0$ solves the moment problem.We can therefore assume that there is a unique measure solving the moment problem for every $b \in \mathfrak{A}$. From [13, Thm 6.10] translated into our GNS-like formulation as in the proof of Proposition 1.6 and using

[4] If $\omega_b$ is not singular, some $\mu^{(a)}_{\omega_b}$ exists due to Proposition 1.6. If it is singular, the zero measure solves the moment problem.

the notation introduced therein, we have that $\mu^{(a)}_{\omega_b}$ is unique if and only if $\pi_{\omega_b}(a)\restriction_{\mathcal{D}_{\omega_b}}$ is essentially selfadjoint in $\mathsf{H}_{\omega_b}$. Now observe that the set of all vectors $\psi_{\omega_b} = \pi_\omega(b)\psi_\omega$, for $b \in \mathfrak{A}$, is dense in $\mathsf{H}_\omega$ since it coincides to $\mathcal{D}_\omega$, so that it is a dense set of uniqueness vectors for the symmetric operator $\pi_\omega(a)$ in $\mathsf{H}_\omega$, which is therefore essentially selfadjoint in view of *Nussbaum's lemma* (Lemma on p. 201 of [14]). Essential selfadjointness of $\pi_{\omega_b}(a)$ can be established similarly, taking (10) and (11) into account. By hypothesis, fixing $b \in \mathfrak{A}$, also the measures $\mu^{(a)}_{\omega_{cb}}$ are uniquely determined by the perturbations $\omega_{cb}$ with $c \in \mathfrak{A}$. The set of all vectors $\psi_{cb} = \pi_\omega(cb)\psi_\omega = \pi_\omega(c)\psi_{\omega_b}$, for $c \in \mathfrak{A}$, is dense in $\mathsf{H}_{\omega_b}$ since it coincides to $\mathcal{D}_{\omega_b}$, so that it is a dense set uniqueness vectors for the symmetric operator $\pi_{\omega_b}(a)$ in $\mathsf{H}_{\omega_b}$ which is essentially selfadjoint due to Nussbaum's lemma again.
(b) Assuming (14), then (13) is a trivial consequence of the uniqueness hypothesis and (a) of Proposition 1.5 applied to the weight $\omega_b$. Let us prove (14) to conclude. We know that $\overline{\pi_\omega(a)}$ admits $\mathcal{D}_{\omega_b}$ as invariant subspace from (10) and (11), furthermore $\pi_\omega(a)\restriction_{\mathcal{D}_{\omega_b}} = \pi_{\omega_b}(a)$ is essentially selfadjoint in the Hilbert space $\mathsf{H}_{\omega_b}$, which is a closed subspace of $\mathsf{H}_\omega$, and $\mathsf{H}_{\omega_b}$ is the closure of $\mathcal{D}_{\omega_b}$. Proposition A.4 implies that $\mathsf{H}_{\omega_b}$ reduces $\overline{\pi_\omega(a)}$ (see Appendix A.1) so that $\overline{\pi_{\omega_b}(a)}$ is the part of $\overline{\pi_\omega(a)}$ on $\mathsf{H}_{\omega_b}$. Using the properties of the PVMs, it is easy to prove that if a closed subspace $\mathsf{H}_0$ reduces a selfadjoint operator $T$, then the PVM of the part of $T$ on $\mathsf{H}_0$ (which is selfadjoint in view of Proposition A.3) is the restriction to $\mathsf{H}_0$ of the PVM of $T$. Applying this result to $\overline{\pi_\omega(a)}$ and $\overline{\pi_{\omega_b}(a)}$, uniqueness of the PVM of a selfadjoint operator implies that the PVM of $\overline{\pi_{\omega_b}(a)}$ is nothing but the restriction to $\mathsf{H}_{\omega_b}$ of the PVM of $\overline{\pi_\omega(a)}$. This is just (14). □

**Example 2.4** The simplest example of application of Theorem 2.3 is that follows. (It is not physically interesting, but it just provides evidence that the very strong hypotheses of Theorem 2.3 are fulfilled in some case.) Consider the unital Abelian $^*$-algebra $\mathbb{C}_{[0,1]}[x]$ made of all complex polynomials in $p : [0,1] \to \mathbb{C}$ in the variable $x$ with the involution defined as the standard point-wise complex conjugation, and the unit given by the constantly 1 polynomial. Define the state

$$\omega(p) = \int_0^1 p(x)dx \quad p \in \mathbb{C}_{[0,1]}[x] \,.$$

Using in particular Stone-Weierstrass theorem and the uniqueness part of the GNS theorem, it is easy to prove that a GNS representation is

$$\mathsf{H}_\omega := L^2([0,1], dx)\,, \quad \pi_\omega(p) = p\cdot\,, \quad \mathcal{D}_\omega = \mathbb{C}_{[0,1]}[x]\,, \quad \psi_\omega = 1$$

where $p\cdot$ denotes the polynomial $p$ acting as multiplicative operator on $\mathbb{C}_{[0,1]}[x]$. Evidently $\omega_q(p) = \int_0^1 |q(x)|^2 p(x)\, dx$. Hence $|\omega_q(x^n)| \leq \frac{C_q}{n+1}$ where $C_q = \max_{[0,1]} |q|^2$. *Carleman's condition* [13, Corollary 4.10] assures that there exists at most one positive Radon measure satisfying (12) for $b = p$ and $a = x$. Hence we can apply Theorem 2.3 and this means in particular that $\pi_\omega(x)$, i.e. the symmetric multiplicative operator $x$ with domain consisting of the complex polynomials on $[0,1]$, is essentially selfadjoint in $L^2([0,1], dx)$. ■

At this point, it may seem plausible that the statement (a) of Proposition 1.5 can be reversed proving that, if $a^* = a \in \mathfrak{A}$, essential selfadjointness of all $\pi_{\omega_b}(a)$ is equivalent to uniqueness of all the measures $\mu^{(a)}_{\omega_b}$. Unfortunately life is not so easy as a consequence of the last item of the example below.

**Example 2.5** Let $\mathfrak{A}_{\mathrm{CCR},1}$ be the one dimensional unital $^*$-algebra which can be realized as follows. $Q, P, I : \mathscr{S}(\mathbb{R}) \to \mathscr{S}(\mathbb{R})$ are respectively the operators

$$(Q\psi)(x) = x\psi(x)\ , \quad (P\psi)(x) = -i\frac{d}{dx}\psi(x)\ , \quad (I\psi)(x) = \psi(x)\ , \quad x \in \mathbb{R}\ .$$

The finite linear combinations of finite compositions of these operators with arbitrary order form a unital $^*$-algebra with unit $I$, provided the involution is defined as $A^* := A^\dagger\restriction_{\mathscr{S}(\mathbb{R})}$ where here $^\dagger$ is the adjoint in $L^2(\mathbb{R}, dx)$, so that $P^* = P$ and $Q^* = Q$. It is important to stress that we are here considering $\mathfrak{A}_{\mathrm{CCR},1}$ as an abstract algebra (i.e., up to isomorphisms of unital $^*$-algebras) independently from the above concrete realization. Consider the state $\omega$

$$\omega(A) = \int_{\mathbb{R}} \overline{\psi_0(x)}(A\psi_0)(x)\, dx \quad A \in \mathfrak{A}_{\mathrm{CCR},1}\ , \quad \psi_0(x) = \pi^{-\frac{1}{4}} e^{-\frac{x^2}{2}}\ .$$

The choice of $\omega$ is evidently related with the ground state of the harmonic oscillator. Exploiting the uniqueness part of the GNS theorem, it is not difficult to prove that the GNS construction generated by $\omega$ leads to

$$\mathsf{H}_\omega = L^2(\mathbb{R}, dx)\ ,\ \pi_\omega(A) = A\restriction_{\mathcal{D}_\omega}\ ,\ \psi_\omega = \psi_0\ .$$

The crucial point which differentiates the found representation of $\mathfrak{A}_{\mathrm{CCR},1}$ form the concrete initial realization, is that now $\mathcal{D}_\omega \subsetneq \mathscr{S}(\mathbb{R})$. Indeed, $\mathcal{D}_\omega$ results to be the dense subspace of $L^2(\mathbb{R}, dx)$ made of all finite linear combinations of *Hermite functions* $\{\psi_n\}_{n\in\mathbb{N}}$ (the eigenstates of the harmonic oscillator Hamiltonian). $\mathcal{D}_\omega$ is dense just because $\{\psi_n\}_{n\in\mathbb{N}}$ is a Hilbert basis of $L^2(\mathbb{R}, dx)$.

Let us pass to consider the perturbations $\omega_B$, $B \in \mathfrak{A}_{\mathrm{CCR},1}$. Using as equivalent generators $I, a, a^*$ (annihilation and creation operators) instead of $I, Q, P$ to define $\mathfrak{A}_{\mathrm{CCR},1}$, one easily sees that

$$\mathcal{D}_\omega = \mathcal{D}_{\omega_B}\ , \quad \mathsf{H}_\omega = \mathsf{H}_{\omega_B}\ , \quad \text{and} \quad \pi_{\omega_B} = \pi_\omega \quad \text{for every choice of } B \in \mathfrak{A}_{\mathrm{CCR},1}\ .$$

The first identity holds because $\psi_\omega \in \mathcal{D}_{\omega_B}$ and $\pi_{\omega_B} = \pi_\omega\restriction_{\mathcal{D}_{\omega_B}}$, for every $B \in \mathfrak{A}_{\mathrm{CCR},1}$ according to (10). The remaining identities are trivial consequences of the first one. Notice that $\pi_{\omega_B}(Q^k)$ (and $\pi_{\omega_B}(P^k)$) are essentially selfadjoint for $k = 1, 2$ because of *Nelson's theorem* [14, Thm X39] as the $\psi_n$s are a set of *analytic vectors* for $\pi_{\omega_B}(Q)$ and $\pi_{\omega_B}(Q^2)$ from the estimate

$$||\pi_\omega(Q^k)^m\psi_n|| \leq 2^{mk/2}\sqrt{(n+mk)!}\,, \qquad m, n = 0, 1, \dots\ , k = 1, 2\ , \tag{15}$$

arising from [14, Example 2 p.204], and their span $\mathcal{D}_{\omega_B} = \mathcal{D}_\omega$ is dense in $\mathsf{H}_{\omega_B} = \mathsf{H}_\omega$. Let us focus on the moment problem relative to $(Q^k, \omega)$. Assume that $s_n^{(k)}$ denotes the $n$-th moment of $Q^k$ in the state $\omega$, namely

$$s_n^{(k)} := \omega(Q^k) = \pi^{-\frac{1}{2}} \int_{\mathbb{R}} x^{kn} e^{-x^2} dx \,. \tag{16}$$

The moment problem relative to $(Q^k, \omega)$ admits at least a solution $\mu_\omega^{(Q^k)}$ due to Proposition 1.6 because $Q^k$ is Hermitian in the algebra. Let us examine uniqueness of this measure, i.e, in the jargon of moment problem theory, we go to check if the moment problem is *determinate* taking advantage of the results discussed in [13].

$k = 1$ We may directly compute $s_n^{(1)} = (2n-1)!!$, which satisfies the hypothesis of *Carleman's condition* [13, Cor. 4.10]: the moment problem for $\{s_n^{(1)}\}_{n\in\mathbb{N}}$ is thus *determinate*.

$k = 2$ We may apply *Cramer's condition* [13, Cor.4.11] to conclude that the moment problem for $\{s_n^{(2)}\}_{n\in\mathbb{N}}$ is again *determinate*.

$k = 3$ We have

$$s_n^{(3)} := \pi^{-\frac{1}{2}} \int_{\mathbb{R}} x^{3n} e^{-x^2} dx = \frac{1}{3\sqrt{\pi}} \int_{\mathbb{R}} y^n \frac{e^{-y^{\frac{2}{3}}}}{y^{\frac{2}{3}}} dy =: \int_{\mathbb{R}} y^n f(y) dy \,. \tag{17}$$

We now apply *Krein's condition* for indeterminacy [13, Thm. 4.14]: since

$$\int_{\mathbb{R}} \frac{\log f(x)}{1+x^2} dx = -\int \frac{\log(3\sqrt{\pi})}{1+x^2} dx - \int_{\mathbb{R}} \frac{x^{\frac{2}{3}}}{1+x^2} dx - \frac{2}{3} \int_{\mathbb{R}} \frac{\log|x|}{1+x^2} dx > -\infty \,,$$

the moment problem for $\{s_n^{(3)}\}_{n\in\mathbb{N}}$ is *not determinate*.

$k = 4$ We have

$$s_n^{(4)} = \frac{1}{\sqrt{\pi}} \int_{\mathbb{R}} x^{4n} e^{-x^2} dx = \frac{2}{\sqrt{\pi}} \int_0^{+\infty} x^{4n} e^{-x^2} dx = \frac{1}{2\sqrt{\pi}} \int_0^{+\infty} y^n \frac{e^{-y^{\frac{1}{2}}}}{y^{\frac{3}{4}}} dy =: \int_{\mathbb{R}} y^n f(y) dy \,.$$

Once again Krein's condition is satisfied:

$$\int_{\mathbb{R}} \frac{\log f(x)}{1+x^2} dx = -\int_0^{+\infty} \frac{\log(2\sqrt{\pi})}{1+x^2} dx - \int_0^{+\infty} \frac{x^{\frac{1}{2}}}{1+x^2} dx - \frac{3}{4} \int_0^{+\infty} \frac{\log x}{1+x^2} dx > -\infty \,.$$

The moment problem associated with $\{s_n^{(4)}\}_{n\in\mathbb{N}}$ is therefore *not determinate*.

The last item furnishes a counter-example to the converse of Theorem 2.3. In fact, there are many measures $\mu_\omega^{(Q^4)}$ associated to the pair $(Q^4, \omega)$ because the moment problem is indeterminate, but every $\pi_{\omega_B}(Q^4) = \pi_\omega(Q^4)$ is essentially selfadjoint. Indeed, $\pi_\omega(Q^4)$ is symmetric, bounded below and a direct computation based on (15) proves that the $\psi_n$ are *semianalytic vectors* for it, hence we can apply [14, Thm X40] which guarantees that $\pi_{\omega_B}(Q^4)$ is essentially selfadjoint. ■

**Remark 2.6** Since $\pi_\omega(Q^4) \geq 0$, we can try to restrict our analysis of existence and uniqueness problem for measures solving the moment problem for the sequence $\{s_n^{(4)}\}_{n\in\mathbb{N}}$ when they are supported in $[0, +\infty)$ rather than in the whole $\mathbb{R}$. This alternate formulation is called *Stieltjes moment problem*. However, if this problem were determinate with unique measure $\mu$, the standard Hamburger problem would be determined as well (but we know that it is not) unless $\mu(\{0\}) \neq 0$ on account of [12, Corollary 8.9]. Since $\overline{\pi_\omega(Q^4)}$ is selfadjoint, that unique $\mu$ would also with the measure (7) obtained from the PVM of $\overline{\pi_\omega(Q^4)}$ with $\mathbb{R}$ replaced for $[0, +\infty)$ since also this spectral measure is a solution of the same moment problem over $[0, +\infty)$. On the other hand, $\overline{\pi_\omega(Q^4)}$ has empty point spectrum (it is the multiplicative operator $x^4$ in $L^2(\mathbb{R}, dx)$), against the assumption $\mu(\{0\}) \neq 0$. Therefore also Stieltjes problem is *not determinate*. ■

Theorem 4.8 below can be in a sense interpreted as a weak converse of Theorem 2.3. However, to see it, a suitable mathematical technology must be introduced.

## 3 The notion of POVM and its relation with symmetric operators

A *Positive Operator Valued Measure* (POVM for short) is an extension of the notion of *Projector Valued Measure* (PVM). Since PVMs are one-to-one with selfadjoint operators and have the physical meaning of a quantum observables (see, e.g., [7] for a wide discussion on the subject), POVMs provide a generalization of the notion of observable. Similarly to the fact that PVMs are related with selfadjoint operators, it results that POVMs are connected to merely *symmetric* operators, even if this interplay is more complicated. Since GNS operators $\pi_\omega(a)$ representing Hermitian elements are in general only symmetric, the notion of POVM seems to be relevant in our discussion on Issue A.

We briefly collect below some material on POVMs and generalized extension of symmetric operator – see [1, 3, 4] for a complete discussion.

**Remark 3.1** The complete equivalence between the notion of POVM used in [3, 4] and the older notion of *spectral function* adopted in [1] is discussed and established in Section 4.9 of [3], especially Theorem 4.3 therein. In [3], spectral functions are called *semispectral functions* while normalized POVMs are named *semispectral measures*. ■

$(\Omega, \Sigma)$ will henceforth denote a measurable space, where $\Sigma$ is a $\sigma$-algebra of sets over $\Omega$. $\mathfrak{B}(\mathsf{H})$ will denote the space of bounded linear operators on the Hilbert space $\mathsf{H}$ and $\mathscr{L}(\mathsf{H}) \subset \mathfrak{B}(\mathsf{H})$ is the space of orthogonal projections over $\mathsf{H}$. We start form the following general definition which admits some other equivalent formulations [4] (see Remark 4.4 of [3] in particular, where the second requirement below is alternatively and equivalently stated).

**Definition 3.2** An operator-valued map $Q\colon \Sigma \to \mathfrak{B}(\mathsf{H})$ is called **positive-operator valued measure** (POVM) if it satisfies the following two conditions:

1. for all $E \in \Sigma$, $Q(E) \geq 0$;

2. for all $\psi, \varphi \in \mathsf{H}$ the map $Q_{\psi,\varphi}\colon \Sigma \ni E \mapsto \langle\psi|Q(E)\varphi\rangle \in \mathbb{C}$ defines a *complex $\sigma$-additive measure*[5] [15].

A POVM $Q$ is said to be **normalized** if $Q(\Omega) = I$. ■

**Remark 3.3** A normalized POVM is a standard PVM (e.g. see [7, 12]) if and only if $Q(E)Q(F) = Q(E \cap F)$ for $E, F \in \Sigma$, so that $Q(E) \in \mathscr{L}(\mathsf{H})$. ■

Since $Q(E) \geq 0$ for every POVM, the map $Q_{\psi,\psi}\colon \Sigma \ni E \mapsto \langle\psi|Q(E)\psi\rangle$ always defines a *finite positive* $\sigma$-additive measure for every fixed $\psi \in \mathsf{H}$ which is also a *probability measure* on $(\Omega, \Sigma)$ if $Q$ is normalized and $||\psi|| = 1$. Similarly to what happens for a PVM, the physical interpretation of $\langle\psi|Q(E)\psi\rangle$ is the probability that, measuring the *generalized observable* associated to the normalized POVM when the state is represented by the normalized vector $\psi$, the outcome belongs to the Borel set $E \subset \mathbb{R}$. What is lost within this more relaxed framework in comparison with the physical interpretation of PVMs (see e.g., [7]) is (a) the logical interpretation of $Q(E)$ as an elementary *YES-NO observable* also known as *test*, (b) the possibility to describe the *post-measurement* state with the standard Lüders-von Neumann reduction postulate exploiting only the POVM (more information must be supplied), (c) the fact that observables $Q(E)$ and $Q(F)$ are necessarily compatible. There exists an extended literature on these topics and we address the reader to [3] for a modern also physically minded treatise on the subject. Another difference concerns the one-to-one correspondence between PVM over $\mathbb{R}$ and selfadjoint operators which, in the standard spectral theory, permits to identify PVMs (quantum observables) with selfadjoint operators. Switching to POVMs, it turns out that there is a more complicated correspondence between normalized POVMs and *symmetric operators* which we will describe shortly.

The typical generalized observable which can be described in terms of a POVM is the (arrival) *time observable* of a particle [3]. That observable cannot be described in terms of selfadjoint operators (PVMs) if one insists on the validity of CCR with respect to the

[5] In particular, the *total variation* of this measure $|Q_{\psi,\varphi}|$ is a *finite* positive $\sigma$-additive measure.

energy observable and these no-go results are popularly known as *Pauli's theorem* (see, e.g. [7, 8]).

A celebrated result due to Naimark establishes that POVMs are connected to PVMs through the famous *Naimark's dilation theorem*, which we state for the case of a normalized POVM [1, Thm. Vol II, p.124] (see [4] for the general case).

**Theorem 3.4** **[Naimark's dilation theorem]** *Let* $Q\colon \Sigma \to \mathfrak{B}(\mathsf{H})$ *be a normalized POVM. Then there exists a Hilbert space* $\mathsf{K}$ *which includes* $\mathsf{H}$ *as a closed subspace, i.e.* $\mathsf{K} = \mathsf{H} \oplus \mathsf{H}^{\perp}$*, and a PVM* $P\colon \Sigma \to \mathscr{L}(\mathsf{K})$ *such that*

$$Q(E) = P_{\mathsf{H}} P(E)\restriction_{\mathsf{H}} \qquad \forall E \in \Sigma\,, \tag{18}$$

*where* $P_{\mathsf{H}} \in \mathscr{L}(\mathsf{K})$ *is the orthogonal projector onto* $\mathsf{H}$*. The triple* $(\mathsf{K}, P_{\mathsf{H}}, P)$ *is called* **Naimark's dilation triple***.*

**Remark 3.5** Another popular way to write (18) for normalized POVMs is

$$Q(E) = V_{\mathsf{H}}^{\dagger} P(E) V_{\mathsf{H}} \qquad \forall E \in \Sigma\,, \tag{19}$$

where $V_{\mathsf{H}} : \mathsf{H} \to \mathsf{K}$ is the embedding isometry, so that $V_{\mathsf{H}} V_{\mathsf{H}}^{\dagger} = I_{\mathsf{H}}$ and $V_{\mathsf{H}} V_{\mathsf{H}}^{\dagger} \in \mathscr{L}(\mathsf{K})$ is the orthogonal projector onto the image of $V_{\mathsf{H}}$, the closed subspace $\mathsf{H}$ of $\mathsf{K}$. In this formulation, *Naimark's dilation triple* is defined as $(\mathsf{K}, V_{\mathsf{H}}, P)$. ■

### *3.2 Generalized selfadjoint extensions of symmetric operators*

POVMs arise naturally when dealing with generalized extensions of *symmetric operators.* As is well known (e.g. see [7, 12]), a selfadjoint operator $A$ in a Hilbert space $\mathsf{H}$ does not admit proper symmetric extensions in $\mathsf{H}$. This is just a case of a more general class of symmetric operators.

**Definition 3.6** A symmetric operator $A$ on a Hilbert space $\mathsf{H}$ is said to be **maximally symmetric** if there is no symmetric operator $B$ on $\mathsf{H}$ such that $B \supsetneq A$. ■

**Remark 3.7**
**(1)** A maximally symmetric operator is necessarily closed, since the closure of a symmetric operator is symmetric as well.
**(2)** It turns out that [1, Thm 3, p.97] *a closed symmetric operator is maximally symmetric (and not selfadjoint) iff one of its deficency indices is* 0 *(and the other does not vanish).*
**(3)** An elementary useful result (immediately arising from, e.g. [7, Thm 5.43]) is that, *if a maximally symmetric operator* $A$ *on* $\mathsf{H}$ *satisfies* $CA \subset AC$ *for a conjugation* $C : \mathsf{H} \to \mathsf{H}$*, then* $A$ *is selfadjoint.* ■

Symmetric operators can also admit extensions in a more general fashion and these extensions play a crucial role in the connection between symmetric operators and POVMs.

**Definition 3.8** Let $A$ be a symmetric operator on a Hilbert space $\mathsf{H}$. A **generalized symmetric** (resp. **selfadjoint**) **extension** of $A$ is a symmetric (resp. selfadjoint) operator on a Hilbert space $\mathsf{K}$ such that

(i) $\mathsf{K}$ contains $\mathsf{H}$ as a closed subspace (possibly $\mathsf{K} = \mathsf{H}$),

(ii) $A \subset B$ in $\mathsf{K}$,

(iii) every closed subspace $\mathsf{K}_0 \subset \mathsf{K}$ such that $\{0\} \neq \mathsf{K}_0 \subset \mathsf{H}^\perp$ does not *reduce* $B$ (see Appendix A.1). ■

Every non-selfadjoint symmetric operator (possibly maximally symmetric) always admits generalized selfadjoint extensions as established in Theorem A.6. Selfadjoint operators are instead maximal also in respect of this more general sort of extension.

**Proposition 3.9** *A selfadjoint operator does not admit proper generalized symmetric extensions.*

*Proof.* See Appendix A.3. □

*3.3 Decomposition of symmetric operators in terms of POVMs*

Naimark extended part of the spectral theory usually formulated in terms of PVMs for normal closed operators (selfadjoint in particular) to the more general case of a symmetric operator [9, 10] where POVMs replace PVMs. A difference with the standard theory is that, unless the symmetric operator is *maximally symmetric*, the POVM which decomposes it is not unique.

**Theorem 3.10** *For a symmetric operator $A$ in the Hilbert space $\mathsf{H}$ the following facts hold.*

**(a)** *There exist a normalized POVM $Q^{(A)} \colon \mathscr{B}(\mathbb{R}) \to \mathfrak{B}(\mathsf{H})$ satisfying*

$$\langle \psi | A\varphi \rangle = \int_{\mathbb{R}} \lambda dQ^{(A)}_{\psi,\varphi}(\lambda) \,, \quad \|A\varphi\|^2 = \int_{\mathbb{R}} \lambda^2 dQ^{(A)}_{\varphi,\varphi}(\lambda) \,, \quad \forall \psi \in \mathsf{H}, \varphi \in D(A) \,, \quad (20)$$

**(b)** *Every normalized POVM $Q^{(A)} \colon \mathscr{B}(\mathbb{R}) \to \mathfrak{B}(\mathsf{H})$ satisfying (20) is of the form*

$$Q^{(A)}(E) := P_{\mathsf{H}} P(E) \restriction_{\mathsf{H}} \qquad \forall E \in \mathscr{B}(\mathbb{R}) \,.$$

*for some Naimark's dilation triple $(\mathsf{K}, P_{\mathsf{H}}, P)$ of $Q^{(A)}$ arising from a generalized selfadjoint extension $B = \int_{\mathbb{R}} \lambda dP(\lambda)$ of $A$ in $\mathsf{K}$,*

$$A = B\restriction_{D(A)\cap\mathsf{H}} \quad \text{and} \quad D(A) \subset \{\psi \in \mathsf{H} | \; \lambda \in L^2(\mathbb{R}, Q^{(A)}_{\psi,\psi})\} \,. \quad (21)$$

**(c)** *A normalized POVM $Q^{(A)}$ satisfying (20) is a PVM if and only if the selfadjoint operator $B$ constructed out of Naimark's dilation triple of $Q^{(A)}$ as in (b) can be chosen as a standard selfadjoint extension of $A$.*

**(d)** *If $A$ is closed, a normalized POVM $Q^{(A)}$ as in (20) exists which, referring to (b), also satisfies*

$$A = B\restriction_{D(B)\cap\mathsf{H}} \quad \textit{and} \quad D(A) = \{\psi \in \mathsf{H} |\ \lambda \in L^2(\mathbb{R}, Q^{(A)}_{\psi,\psi})\}\ . \qquad (22)$$

**(e)** *$A$ is maximally symmetric if and only if there is a unique normalized POVM $Q^{(A)}$ as in (20). In this case, (22) is valid for all choices of $(\mathsf{K}, P_{\mathsf{H}}, P)$ generating $Q^{(A)}$ as in (b).*

**(f)** *If $A$ is selfadjoint, there is a unique normalized POVM $Q^{(A)}$ satisfying (20), and it is a PVM. In this case $\mathsf{K} = \mathsf{H}$, $Q^{(A)} = P$, and $A = B$ for all choices of $(\mathsf{K}, P_{\mathsf{H}}, P)$ generating $Q^{(A)}$ as in (b).*

*Proof.* See Appendix A.3. □

**Corollary 3.11** *Let $A$ be a symmetric operator in $\mathsf{H}$. The following facts are true.*

**(a)** *$A$ and $\overline{A}$ admits the same class of POVMs satisfying (a) of Theorem 3.10 for $A$ and $\overline{A}$ respectively.*

**(b)** *$A$ admits a unique normalized POVM as in (a) of Theorem 3.10 if and only $\overline{A}$ is maximally symmetric. In this case*

$$D(\overline{A}) = \{\psi \in \mathsf{H} |\ \lambda \in L^2(\mathbb{R}, Q^{(A)}_{\psi,\psi})\}\ .$$

**(c)** *The unique normalized POVM as in (b) is a PVM if $A$ is also essentially selfadjoint.*

*Proof.* Every generalized selfadjoint extension of $\overline{A}$ is a generalized extension of $A$, since $A \subset \overline{A}$. Every generalized selfadjoint extension of $A$ is closed (because selfadjoint) so that it is also a generalized selfadjoint extension of $\overline{A}$. In view of (b) of Theorem 3.10, $A$ and $\overline{A}$ have the same class of associated POVMs satisfying (a) of the that theorem. Therefore $A$ admits a unique POVM if and only if $\overline{A}$ is maximally symmetric as a consequence of (e) and the identity regarding $D(\overline{A})$ is valid in view of (d) of Theorem 3.10. Finally, this POVM is a PVM if $A$ is also essentially selfadjoint due to (f) Theorem 3.10. □

**Definition 3.12** If $A$ is symmetric operator in the Hilbert space $\mathsf{H}$, a normalized POVM $Q^{(A)}$ over the Borel algebra over $\mathbb{R}$ which satisfies (a) of Theorem 3.10, i.e.

$$\langle\psi|A\varphi\rangle = \int_{\mathbb{R}} \lambda dQ^{(A)}_{\psi,\varphi}(\lambda)\,, \qquad ||A\varphi||^2 = \int_{\mathbb{R}} \lambda^2 dQ^{(A)}_{\varphi,\varphi}(\lambda)\,, \qquad \forall\psi \in \mathsf{H}, \varphi \in D(A)\,,$$

is said to be **associated** to $A$ or, equivalently, to **decompose** $A$. ■

*3.4 Hermitian operators as integrals of POVMs*

While a symmetric operator admits at least one normalized POVM which decomposes it according to Definition 3.12, not all normalized POVM decomposes symmetric operators. The main obstruction comes from the second equation in (20) as well as from the difficulty to identify a convenient notion of operator integral with respect to a POVM. This aspect of POVMs has been investigated in [4] (see also [3] for further physical comments) in wide generality. We only state and prove an elementary result which, though it is not explicitly stated in [4], it is however part of the results discussed therein. In particular, every POVM over $\mathbb{R}$ can be weakly integrated determining a *unique* Hermitian operator over a natural domain. It is worth stressing that the result strictly depends on the choice of this domain and different alternatives are possible in principle [3, 4].

**Theorem 3.13** *If $Q\colon \mathscr{B}(\mathbb{R}) \to \mathfrak{B}(\mathsf{H})$ is a normalized POVM in the Hilbert space $\mathsf{H}$, define the subset $D(A^{(Q)}) \subset \mathsf{H}$,*

$$D(A^{(Q)}) := \left\{ \psi \in \mathsf{H} \,\middle|\, \int_{\mathbb{R}} \lambda^2 dQ_{\psi,\psi}(\lambda) < +\infty \right\} . \tag{23}$$

*The following facts are valid.*

- **(a)** *$D(A^{(Q)})$ is a subspace of $\mathsf{H}$ (which is not necessarily dense or non-trivial).*
- **(b)** *There exists a unique operator $A^{(Q)} : D(A^{(Q)}) \to \mathsf{H}$ such that*

$$\langle \varphi | A^{(Q)} \psi \rangle = \int_{\mathbb{R}} \lambda dQ_{\varphi,\psi}(\lambda) \,, \quad \forall \varphi \in \mathsf{H} \,, \forall \psi \in D(A^{(Q)}) \,. \tag{24}$$

- **(c)** *$A^{(Q)}$ is Hermitian, so that $A^{(Q)}$ is symmetric if and only if $D(A^{(Q)})$ is dense.*
- **(d)** *If $(\mathsf{K}, P_{\mathsf{H}}, P)$ is a Naimark's dilation triple of $Q$, then*

$$A\psi = P_{\mathsf{H}} \int_{\mathbb{R}} \lambda dP(\lambda)\psi \,, \quad \forall \psi \in D(A^{(Q)}) \,. \tag{25}$$

- **(e)** *If there exists a Naimark's dilation triple $(\mathsf{K}, P_{\mathsf{H}}, P)$ of $Q$ such that*

$$\int_{\mathbb{R}} \lambda dP(\lambda)(D(A^{(Q)})) \subset \mathsf{H} \,,$$

*then $A$ is closed and*

$$||A^{(Q)}\psi||^2 = \int_{\mathbb{R}} \lambda^2 dQ_{\psi,\psi}(\lambda) \,, \quad \forall \psi \in D(A^{(Q)}) \,. \tag{26}$$

*Proof.* See Appendix A.3. □

**Remark 3.14** Theorems 3.10 and 3.13 can be used to define a function $f(A)$ of a symmetric operator $A$ in $\mathsf{H}$ when $A$ itself can be decomposed along the normalized POVM $Q^{(A)}$ according to Definition 3.12, and $f : \mathbb{R} \to \mathbb{R}$ is Borel measurable. It is symply sufficient observing that $Q'(E) := Q^{(A)}(f^{-1}(E))$ is still a normalized POVM when $E$ varies in $\mathscr{B}(\mathbb{R})$, so that $f(A)$ can be defined according to definitions (23) and (24) just by integrating $Q'$. Therefore, from the standard measure theory, it arises

$$\langle \varphi | f(A)\psi \rangle = \int_{\mathbb{R}} \mu \, dQ'_{\varphi,\psi}(\mu) = \int_{\mathbb{R}} f(\lambda) \, dQ^{(A)}_{\varphi,\psi}(\lambda) \quad \text{if } \varphi \in \mathsf{H} \text{ and } \psi \in D(f(A)) \,, \tag{27}$$

$$D(f(A)) = \left\{ \psi \in \mathsf{H} \;\middle|\; \int_{\mathbb{R}} \mu^2 dQ'_{\psi,\psi}(\mu) < +\infty \right\} = \left\{ \psi \in \mathsf{H} \;\middle|\; \int_{\mathbb{R}} |f(\lambda)|^2 dQ^{(A)}_{\psi,\psi}(\lambda) < +\infty \right\} . \tag{28}$$

When $A$ is selfadjoint, so that we deal with a PVM, this definition of $f(A)$ coincides to the standard one. It is however necessary to stress that, when $Q$ is properly a POVM,

(a) unless $f$ is bounded (in that case $D(f(A)) = \mathsf{H}$), there is no guarantee that the Hermitian operator $f(A)$ has a dense domain nor that[6] $||f(A)\psi||^2 = \int_{\mathbb{R}} |f(\lambda)|^2 dQ^{(A)}_{\psi,\psi}(\lambda)$ for $\psi \in D(f(A))$,

(b) $f(A)$ does not enjoy the same properties as those of the standard functional calculus of selfadjoint operators (see, e.g. [7]), just because the fundamental property of PVMs $Q(E)Q(E') = Q(E \cap E')$ is false for POVMs,

(c) the notion of $f(A)$ also depends on the normalized POVM $Q^{(A)}$ exploited to decompose $A$, since $Q^{(A)}$ is unique if and only if $A$ is maximally symmetric. ■

# 4 Generalized notion of observable $\pi_\omega(a)$ and expectation-value intepretation of $\omega(a)$

We are in a position to apply the developed theory to tackle the initial problems stated in issues A and B establishing the main results of this work.

## *4.1 Generalized observable $\pi_\omega(a)$ and POVMs*

Coming back to symmetric operators arising from GNS representations, the summarized theory of POVMs and Corollary 3.11 in particular have some important consequences concerning the interpretation of $\pi_\omega(a)$ as an observable, when it is not essentially selfadjoint. Consider the symmetric operator $\pi_\omega(a)$ when $a^* = a \in \mathfrak{A}$ and $\omega : \mathfrak{A} \to \mathbb{C}$ is a finite weight on the unital $^*$-algebra $\mathfrak{A}$. We have that

[6] If $f : \mathbb{R} \ni \lambda \to \lambda \in \mathbb{R}$, we have $A = f(A)\restriction_{D(A)}$, but the domain of $f(A)$ according to (23) is in general larger than $D(A)$, and $||f(A)\psi||^2 = \int_{\mathbb{R}} |f(\lambda)|^2 dQ^{(A)}_{\psi,\psi}(\lambda)$ is valid for $\psi \in D(A)$.

(1) $\pi_\omega(a)$ and $\overline{\pi_\omega(a)}$ share the same class of associated normalized POVMs $Q^{(a,\omega)}$ so that they support the same physical information when interpreting them as generalized observables. More precisely, each of these POVMs endows those symmetric operators with the physical meaning of generalized observable in the Hilbert space $\mathsf{H}_\omega$. This is particularly relevant when $\pi_\omega(a)$ does not admit selfadjoint extensions;

(2) the above class of normalized POVMs however includes also all possible PVMs of all possible selfadjoint extensions of $\pi_\omega(a)$ if any. Hence, the standard meaning of observable in Hilbert space is encompassed;

(3) $Q^{(a,\omega)}$ is unique if and only if $\overline{\pi_\omega(a)}$ is maximally symmetric but not necessarily selfadjoint;

(4) That unique POVM is a PVM if $\pi_\omega(a)$ is essentially selfadjoint.

Even if the operator $\pi_\omega(a)$ does *not* admit a selfadjoint extension, it can be considered a generalized observable, since it admits decompositions in terms of POVMs which are generalized observables in their own right.

### *4.2 Expectation-value intepretation of $\omega_b(a)$ by means of consistent class of measures solving the moment problem*

Let us now come to the expectation-value intepretation of $\omega(a)$ extended to the perturbations $\omega_b(a)$. This inteprentation relies upon the choice of a measure $\mu^{(a)}_\omega$ viewed as particular cases of the large class of measures $\mu^{(a)}_{\omega_b}$ associated to perturbed states $\omega_b$. All these measures are assumed to solve the moment problem (12) for $(a, \omega_b)$, where the case $n = 1$ is just the expectation-value intepretation of $\omega_b(a)$ and $\omega(a)$ in particular for $b = \mathbb{I}$.

The last item in Example 2.5 shows that there are many measures solving the moment problem relative to $(a, \omega_b)$ in general, even if the operator $\pi_\omega(a)$ is essentially selfadjoint. We need some physically meaningful strategy to reduce the number of those measures.

This section proves that, once we have imposed suitable physically meaningful requirements on the measures $\mu^{(a)}_{\omega_b}$, a new connection arises between the remaining classes of physically meaningful measures and POVMs decomposing the symmetric operators $\pi_{\omega_b}(a)$. These POVMs also generate the said measures $\mu^{(a)}_{\omega_b}$.

We start by noticing that when $b$ is a function of the Hermitian element $a \in \mathfrak{A}$, the measures $\mu^{(a)}_{\omega_b}$ ($k \in \mathbb{N}$) and $\mu^{(a)}_\omega$ are not independent, in particular it holds

$$\int_{\mathbb{R}} \lambda^{2k+1} d\mu^{(a)}_\omega(\lambda) = \omega(a^{2k+1}) = \omega_{a^{2k}}(a) = \int_{\mathbb{R}} \lambda \, d\mu^{(a)}_{\omega_{a^{2k}}}(\lambda)\,.$$

However, referring only to the subalgebra generated by $a$, we miss the information of the whole algebra $\mathfrak{A}$ which contains $a$. We therefore try to restrict the class of the measure $\mu^{(a)}_{\omega_b}$ by imposing some natural compatibility conditions among the measures $\mu^{(a)}_{\omega_b}$ associated with *completely general* elements $b \in \mathfrak{A}$. As a starting point, let us consider a family of measure $\{\mu^{(a)}_{\omega_b}\}_{b\in\mathfrak{A}}$ each of which is a solution to the moment problem (12) relative to $(a,\omega_b)$, with $\mu^{(a)}_{\omega_b} = 0$ if $\omega_b$ is singular. Since for all $b, c \in \mathfrak{A}$, $z \in \mathbb{C}$, and every real polynomial $p$,

$$\omega_{b+c}(p(a)) + \omega_{b-c}(p(a)) = 2[\omega_b(p(a)) + \omega_c(p(a))]\,.$$
$$\omega_{zb}(p(a)) = |z|^2\omega_b(p(a))\,, \tag{29}$$

we also have

$$\int_{\mathbb{R}} p(\lambda)d\mu^{(a)}_{\omega_{b+c}}(\lambda) + \int_{\mathbb{R}} p(\lambda)d\mu^{(a)}_{\omega_{b-c}}(\lambda) = 2\left[\int_{\mathbb{R}} p(\lambda)d\mu^{(a)}_{\omega_b}(\lambda) + \int_{\mathbb{R}} p(\lambda)d\mu^{(a)}_{\omega_c}(\lambda)\right]. \tag{30}$$
$$\int_{\mathbb{R}} p(\lambda)d\mu^{(a)}_{\omega_{zb}}(\lambda) = |z|^2\int_{\mathbb{R}} p(\lambda)d\mu^{(a)}_{\omega_b}(\lambda)\,, \tag{31}$$

Finally observe that the following directional continuity property holds true for $b, c, a = a^* \in \mathfrak{A}$, and every real polynomial $p$,

$$\omega_{b+tc}(p(a)) \to \omega_b(p(a)) \quad \text{for } \mathbb{R} \ni t \to 0\,,$$

that implies

$$\int_{\mathbb{R}} p(\lambda)d\mu^{(a)}_{\omega_{b+tc}}(\lambda) \to \int_{\mathbb{R}} p(\lambda)d\mu^{(a)}_{\omega_b}(\lambda) \quad \text{for } \mathbb{R} \ni t \to 0\,. \tag{32}$$

Identities (30)-(32) are true for every choice of measures associated with the algebraic observable $a$ and the perturbations $\omega_b$, so that they *cannot* be used as contraints to reduce the number of those measures.

We observe that the above relations actually regard *polynomials* $p(a)$ of $a$. From the physical side, dealing only with polynomials seems a limitation since we expect that, at the end of game, after having introduced some technical information, one would be able to define more complicated functions of $a$ (as it happens when dealing with $C^*$-algebras), because these observables are physically necessary and have a straightforward operational definition: *$f(a)$ is the observable which attains the values $f(\lambda)$, where $\lambda$ are the values attained by $a$.* We restrict ourselves to bounded functions to avoid subtleties with domains. If $\mu^{(a)}_{\omega_b}$ is physically meaningful and $f : \mathbb{R} \to \mathbb{R}$ is a bounded measurable function, we expect that the (unknown) observable $f(a)$ is however represented by the function $f(\lambda)$ in the space $L^2(\mathbb{R}, d\mu^{(a)}_{\omega_b})$ and, as far as expectation values are concerned, $\omega_b(f(a)) = \int_{\mathbb{R}} f(\lambda)\, d\mu^{(a)}_{\omega_b}(\lambda)$.

This viewpoint can be also heuristically supported from another side. If we deal with $\pi_{\omega_b}(a)$ instead of $a$ itself and we decompose the symmetric operator $\pi_{\omega_b}(a)$ with a POVM, the function $f(\pi_{\omega_b}(a))$ can be defined according to Remark 3.14. If we now assume that $\mu^{(a)}_{\omega_b} = Q^{(\pi_{\omega_b}(a))}_{\psi_b,\psi_b}$ we just have that the abstract observable $f(a)$ is represented by the function $f(\lambda)$ when we compute the expectation values: according to (27) for $\varphi = \psi = \psi_b$, we have $\omega_b(f(a)) = \langle\psi_b|\pi_{\omega_b}(a)\psi_b\rangle = \int_{\mathbb{R}} f(\lambda)\, d\mu^{(a)}_{\omega_b}(\lambda)$.

We therefore strengthen equations (30)-(32) *by requiring that the physically interesting measures are such that (30)-(32) are valid for arbitrary bounded measurable functions* $f : \mathbb{R} \to \mathbb{R}$ *in place of polynomials* $p$.
The resulting condition, just specializing to characteristic functions $f = \chi_E$ for every Borel measurable set $E$ over the real line, leads to the following identities, which imply the previous ones (stated for general bounded measurable functions)

$$\mu^{(a)}_{\omega_{b+c}} + \mu^{(a)}_{\omega_{b-c}} = 2\big[\mu^{(a)}_{\omega_b} + \mu^{(a)}_{\omega_c}\big] \,, \quad \mu^{(a)}_{\omega_{zb}} = |z|^2 \mu^{(a)}_{\omega_b} \tag{33}$$

$$\mu^{(a)}_{\omega_{b+tc}}(E) \to \mu^{(a)}_{\omega_b}(E) \quad \text{if } \mathbb{R} \ni t \to 0, \tag{34}$$

**Remark 4.1**
**(1)** We stress that (33) and (34) are *not* consequences of (30)-(31) in the general case, in particular because polynomials are not necessarily dense in the relevant $L^1$ spaces, since the considered Borel measures have not compact support in general and we cannot directly apply Stone-Weierstass theorem. (33) and (34) are however necessarily satisfied when all considered measures $\mu^{(a)}_{\omega_b}$ are induced by a unique PVM as for instance in the strong hypotheses of Theorem 2.3: (13) immediately implies (33) and (34). This is also the case for a $C^*$-algebra, since the measures arise from PVMs due to Proposition 1.5.
**(2)** Identities (33) and (34) remain valid also when labeling the measures with the classes $[b] \in \mathfrak{A}/G(\mathfrak{A}, \omega) = \mathcal{D}_\omega$ since these only involve the linear structure of $\mathfrak{A}$ which survive the quotient operation. ■

We can state the following general definition, taking remark (2) into account in particular.

**Definition 4.2** If $\mathcal{D}$ is a complex vector space, a family of positive $\sigma$-additive measures $\{\nu_\psi\}_{\psi \in \mathcal{D}}$ over the measurable space $(\Omega, \Sigma)$ such that

$$\nu_{\psi+\varphi} + \nu_{\psi-\varphi} = 2\big[\nu_\psi + \nu_\varphi\big] \,, \quad \nu_{z\psi} = |z|^2 \nu_\psi \qquad \text{for all } \psi, \varphi \in \mathcal{D} \text{ and } z \in \mathbb{C} \tag{35}$$

$$\nu_{\psi+t\varphi}(E) \to \nu_\psi(E) \quad \text{if } \mathbb{R} \ni t \to 0, \quad \text{for every fixed triple } \psi, \varphi \in \mathcal{D} \text{ and } E \in \Sigma. \tag{36}$$

is said to be **consistent**. ■

**Remark 4.3** From Definition 4.2, $\nu_0$ is the zero measure ($\nu_0(E) = 0$ if $E \in \Sigma$). ■

### *4.3 Consistent classes of measures and POVMs*

We now apply the summarized theory of POVMs to prove that the family of POVMs associated to $\pi_\omega(a)$ is one-to-one with the family of consistent classes of measures solving the moment problem for all $\omega_b$. The proof consists of two steps. Here is the former.

If $Q^{(a,\omega)}$ is a POVM associated to $\pi_\omega(a)$ for $a^* = a \in \mathfrak{A}$ and for a finite weight $\omega : \mathfrak{A} \to \mathbb{C}$, let $\nu^{(a)}_{\omega_b}$ be the Borel measure defined by

$$\nu^{(a)}_{\omega_b}(E) := \langle \psi_b | Q^{(a,\omega)}(E) \psi_b \rangle \quad \text{if } E \in \mathscr{B}(\mathbb{R}) \,, \tag{37}$$

for every perturbation $\omega_b$.

**Theorem 4.4** *Consider the unital* $^*$*-algebra* $\mathfrak{A}$*, a weight* $\omega : \mathfrak{A} \to \mathbb{C}$*, an element* $a = a^* \in \mathfrak{A}$ *and the family of measures* $\{\nu^{(a)}_{\omega_b}\}_{b\in\mathfrak{A}}$ *defined in (37) with respect to a normalized POVM* $Q^{(a,\omega)}$ *associated to* $\pi_\omega(a)$*. The following facts are true.*

**(a)** $\{\nu^{(a)}_{\omega_b}\}_{b\in\mathfrak{A}}$ *is a consistent family over* $\mathcal{D}_\omega = \mathfrak{A}/G(\mathfrak{A},\omega)$*.*

**(b)** *Each* $\nu^{(a)}_{\omega_b}$ *is a solution of the moment problem (12) relative to* $(a, \omega_b)$*.*

*Proof.* Let us focus on Theorem 3.10 for $A := \pi_\omega(a)$ with $D(A) = \mathcal{D}_\omega$ and $\mathsf{H} = \mathsf{H}_\omega$. According to (b), the POVM $Q^{(A)} = Q^{(a,\omega)}$ can be written as $Q^{(A)} = P_{\mathsf{H}}P$ for a PVM $P$ of a selfadjoint operator $B : D(B) \to \mathsf{K}$ defined on a larger Hilbert space $\mathsf{K}$, including $\mathsf{H}$ as a closed subspace, such that $A = B\restriction_{D(A)\cap\mathsf{H}}$. Observe that $\pi_\omega(b)\psi_\omega \in \mathcal{D}_\omega = D(\pi_\omega(a^n)) = D(\pi_\omega(a)^n) = D(A^n) \subset D(B^n)$ where, in the last inclusion, we have exploited $A = B\restriction_{D(A)\cap\mathsf{H}}$ and $A(\mathcal{D}_\omega) \subset \mathcal{D}_\omega$. By the standard spectral theory of selfadjoint operators (see, e.g., [7]) we therefore have ($\psi_b := \pi_\omega(b)\psi_\omega$)

$$\langle \pi_\omega(b)\psi_\omega | B^n \pi_\omega(b)\psi_\omega\rangle = \int_{\mathbb{R}} \lambda^n dP_{\psi_b,\psi_b}(\lambda) = \int_{\mathbb{R}} \lambda^n dQ^{(A)}_{\psi_b,\psi_b}(\lambda) = \int_{\mathbb{R}} \lambda^n d\nu^{(a)}_{\omega_b}(\lambda)\,,$$

where, in the last passage we have used $Q^{(A)} = P_{\mathsf{H}}P$, $P_{\mathsf{H}}\pi_\omega(b)\psi_\omega = \pi_\omega(b)\psi_\omega$, and (37). On the other hand, per construction, $A^n\pi_\omega(b)\psi_\omega = B^n\pi_\omega(b)\psi_\omega$ and eventually the GNS theorem yields $\omega_b(a^n) = \langle \pi_\omega(b)\psi_\omega | A^n\pi_\omega(b)\psi_\omega\rangle = \langle \pi_\omega(b)\psi_\omega | B^n\pi_\omega(b)\psi_\omega\rangle$. In summary, if $n = 0, 1, 2\ldots$ and $b \in \mathfrak{A}$,

$$\omega_b(a^n) = \int_{\mathbb{R}} \lambda^n dQ^{(A)}_{\psi_b,\psi_b}(\lambda)\,.$$

We have established that each measure (37) is a solutions of the moment problem relative to $(a, \omega_b)$. By direct inspection, one immediately sees that $\{\nu^{(a)}_{\omega_b}\}_{b\in\mathfrak{A}}$ defined as in (37) satisfies Definition 4.2. $\square$

The result is reversed with the help of the following abstract technical proposition.

**Proposition 4.5** *Let* $X$ *be a complex vector space and* $p : X \to [0, +\infty)$ *such that*

*(i)* $p(\lambda x) = |\lambda| p(x)$ *for every pair* $x \in X$ *and* $\lambda \in \mathbb{C}$*,*

*(ii)* $p(x + y)^2 + p(x - y)^2 = 2[p(x)^2 + p(y)^2]$ *for every pair* $x, y \in X$*,*

*(iii)* $p(x + ty) \to p(x)$ *for* $\mathbb{R} \ni t \to 0^+$ *and every fixed pair* $x, y \in X$*.*

*Under these hypotheses,*

**(a)** *$p$ is a seminorm on* $X$*,*

**(b)** *there is a unique positive semi definite Hermitian scalar product* $X \times X \ni (x, y) \mapsto (x|y)_p \in \mathbb{C}$ *such that* $p(x) = \sqrt{(x|x)_p}$ *for all* $x \in X$*,*

**(c)** *the scalar product in (b) satisfies*

$$(x|y)_p = \frac{1}{4}\sum_{k=0}^{3}(-i)^k p(x+i^k y)^2 \qquad \text{for } x, y \in X. \tag{38}$$

*Proof.* See Appendix A.3 $\square$

We can now establish another main result of the work, which is the converse of Theorem 4.4. Together with the afore-mentioned theorem, it proves that for $a^* = a \in \mathfrak{A}$ and a finite weight $\omega : \mathfrak{A} \to \mathbb{C}$, the family of normalized POVMs associated with the given symmetric operator $\pi_\omega(a)$ is one-to-one with the family of consistent classes of measures of all $\omega_b(a)$ which solve the moment problem for all perturbations $\omega_b$, when $b \in \mathfrak{A}$.

**Theorem 4.6** *Consider the unital* $^*$*-algebra* $\mathfrak{A}$*, a finite weight* $\omega : \mathfrak{A} \to \mathbb{C}$*,* $a = a^* \in \mathfrak{A}$*, and a consistent class of measures* $\{\mu^{(a)}_{\omega_b}\}_{b\in\mathfrak{A}}$ *solutions of the moment problem relative to the pairs* $(a, \omega_b)$ *for* $b \in \mathfrak{A}$*. The following facts are true.*

**(a)** *There is a unique normalized POVM* $Q^{(a,\omega)} : \mathscr{B}(\mathbb{R}) \to \mathcal{B}(\mathsf{H}_\omega)$ *such that, if* $b \in \mathfrak{A}$*,*

$$\mu^{(a)}_{\omega_b}(E) = \langle \psi_b | Q^{(a,\omega)}(E)\psi_b \rangle \qquad \forall E \in \mathscr{B}(\mathbb{R})\,, \tag{39}$$

**(b)** $Q^{(a,\omega)}$ *decomposes* $\pi_\omega(a)$ *according to Definition 3.12 so that, in particular,*

$$\mathcal{D}_\omega = D(\pi_\omega(a)) \subset \left\{ \psi \in \mathsf{H}_\omega \;\middle|\; \int_{\mathbb{R}} \lambda^2 dQ^{(a,\omega)}_{\psi,\psi}(\lambda) < +\infty \right\}\,. \tag{40}$$

**(c)** $Q^{(a,\omega)}$ *is unique if and only if* $\overline{\pi_\omega(a)}$ *is maximally symmetric. In this case*

$$D(\overline{\pi_\omega(a)}) = \left\{ \psi \in \mathsf{H}_\omega \;\middle|\; \int_{\mathbb{R}} \lambda^2 dQ^{(a,\omega)}_{\psi,\psi}(\lambda) < +\infty \right\} \tag{41}$$

*and that unique* $Q^{(a,\omega)}$ *is a PVM if and only if* $\overline{\pi_\omega(a)}$ *is selfadjoint. In that case* $Q^{(a,\omega)}$ *coincides with the PVM of* $\overline{\pi_\omega(a)}$*.*

**(d)** *Let us define*

$$Q^{(a,\omega_b)}(E) := P_{\omega_b} Q^{(a,\omega)}(E)\restriction_{\mathsf{H}_{\omega_b}}\,, \tag{42}$$

*where* $P_{\omega_b} : \mathsf{H}_\omega \to \mathsf{H}_\omega$ *is the orthogonal projector onto* $\mathsf{H}_{\omega_b}$*. It turns out that, for* $b \in \mathfrak{A}$*,*

*(i)* $Q^{(a,\omega_b)}$ *is a normalized POVM in* $\mathsf{H}_{\omega_b}$*.*

*(ii) It holds*

$$\langle \psi_b | Q^{(a,\omega_b)}(E)\psi_b \rangle = \mu^{(a)}_{\omega_b}(E) \qquad \forall E \in \mathscr{B}(\mathbb{R})\,, \tag{43}$$

*(iii)* $Q^{(a,\omega_b)}$ *decomposes* $\pi_{\omega_b}(a)$ *in the sense of Definition 3.12.*

*Proof.* (a) It is clear that, if a normalized POVM exists satisfying (39) for all $b \in \mathfrak{A}$, then it is unique. Indeed, taking advantage of the polarization identity, another similar POVM $Q$ would satisfy $\langle\psi_b|(Q(E)-Q^{(a,\omega)}(E))\psi_c\rangle = 0$ for every $\psi_b, \psi_c \in \mathcal{D}_\omega$, which is a dense set. Therefore, $(Q(E) - Q^{(a,\omega)}(E))\psi_c = 0$ for every $\psi_c \in \mathcal{D}_\omega$. Continuity of $Q(E) - Q^{(a,\omega)}(E)$ yields $Q(E) = Q^{(a,\omega)}(E)$.
Let us prove that a normalized POVM satisfying (39) for all $b \in \mathfrak{A}$ exists. Fix $E \in \mathscr{B}(\mathbb{R})$. Since the positive measures $\nu_{\psi_b} := \mu^{(a)}_{\omega_b}$ satisfy the identities (35) and (36), $\mathcal{D}_\omega \ni \psi \mapsto \nu_\psi(E)$ fulfills the hypotheses of Proposition 4.5. Consequently, that function is a seminorm over $\mathcal{D}_\omega$ and there is a unique semidefinite Hermitian scalar product inducing it:

$$(\psi_b|\psi_c)_E := \frac{1}{4}\sum_{k=0}^{3}(-i)^k \mu^{(a)}_{\omega_{b+i^k c}}(E)\,. \tag{44}$$

Applying Cauchy-Schwartz inequality, we have

$$|(\psi_b|\psi_c)_E \leq \|b\|^2_E\|c\|^2_E = \mu^{(a)}_{\omega_b}(E)\mu^{(a)}_{\omega_c}(E) \leq \mu^{(a)}_{\omega_b}(\mathbb{R})\mu^{(a)}_{\omega_c}(\mathbb{R}) = \omega_b(\mathbb{I})\omega_c(\mathbb{I}) = ||\psi_b||^2_{\mathsf{H}_\omega}\,||\psi_c||^2_{\mathsf{H}_\omega}\,.$$

Exploiting Riesz' theorem, it then follows that $(\,|\,)_E$ continuously extends to $\mathsf{H}_\omega \times \mathsf{H}_\omega$ and moreover there exists a unique selfadjoint positive operator $Q(E) \in \mathfrak{B}(\mathsf{H}_\omega)$ with $0 \leq Q(E) \leq 1$ such that

$$(\psi|\varphi)_E = \langle\psi|Q(E)\varphi\rangle \qquad \forall\psi,\varphi \in \mathsf{H}_\omega\,. \tag{45}$$

The map $Q\colon \mathscr{B}(\mathbb{R}) \ni E \mapsto Q(E) \in \mathfrak{B}(\mathsf{H}_\omega)$ is a normalized POVM according to Definition 3.2 as we go to prove. In fact, $Q(E) \geq 0$ as said above and, since

$$\langle\psi_b|Q(E)\psi_c\rangle = \frac{1}{4}\sum_{k=0}^{3}(-i)^k \mu^{(a)}_{\omega_{b+i^k c}}(E) \qquad \forall\psi_b,\psi_c \in \mathcal{D}_\omega\,, \tag{46}$$

the left-hand side is a complex Borel measure over $\mathbb{R}$ since the right-hand side is a complex combination of such measures. Finally, $E = \mathbb{R}$ produces

$$\langle\psi_b|Q(\mathbb{R})\psi_c\rangle = \frac{1}{4}\sum_{k=0}^{3}(-i)^k||\psi_b + ik\psi_c||^2 = \langle\psi_b|\psi_c\rangle \qquad \forall\psi_b,\psi_c \in \mathcal{D}_\omega\,. \tag{47}$$

As $\mathcal{D}_\omega$ is dense in $\mathsf{H}_\omega$, it implies $Q(\mathbb{R}) = I$ so that the candidate POVM $Q$ is normalised. To conclude the proof of the fact that $Q^{(a,\omega)} := Q$ is a POVM, it is sufficient to prove that $\mathscr{B}(\mathbb{R}) \ni E \mapsto \langle\psi|Q(E)\varphi\rangle \in \mathbb{C}$ is a complex measure no matter we choose $\psi,\phi \in \mathsf{H}_\omega$ (and not only for $\psi,\phi \in \mathcal{D}_\omega$ as we already know). A continuity argument from the case of $\psi,\varphi \in \mathcal{D}_\omega$ proves that the said map is at least additive so that, in particular $\langle\psi|Q(\varnothing)\varphi\rangle = 0$, because $Q(\mathbb{R}) = I$. Let us pass to prove that the considered function is unconditionally $\sigma$-additive so that it is a complex measure as wanted. If the sets $E_n \in \mathscr{B}(\mathbb{R})$ when $n \in \mathbb{N}$ satisfy $E_k \cap E_h = \varnothing$ for $h \neq k$, consider the difference

$$\Delta_N := \sum_{n=0}^{N}\langle\psi|Q(E_n)\varphi\rangle - \langle\psi|Q(E)\varphi\rangle$$

where $E := \cup_{n\in\mathbb{N}} E_n$. We want to prove that $\Delta_N \to 0$ for $N \to +\infty$. $\Delta_N$ can be decomposed as follows

$$\Delta_N = \sum_{n=0}^{N}\langle\psi-\psi_b|Q(E_n)(\varphi-\psi_c)\rangle - \sum_{n=0}^{N}\langle\psi_b|Q(E_n)(\varphi-\psi_c)\rangle - \sum_{n=0}^{N}\langle\psi-\psi_b|Q(E_n)\psi_c\rangle$$
$$+\sum_{n=0}^{N}\langle\psi_b|Q(E_n)\psi_c\rangle - \langle\psi_b|Q(E)\psi_c\rangle$$
$$-\langle\psi-\psi_b|Q(E)(\varphi-\psi_c)\rangle + \langle\psi_b|Q(E)(\varphi-\psi_c)\rangle + \langle\psi-\psi_b|Q(E)\psi_c\rangle\,.$$

Using additivity and defining $F_N := \cup_{n=0}^N E_n$, we can re-arrange the found expansion as

$$\Delta_N = \langle\psi-\psi_b|Q(F_N)(\varphi-\psi_c)\rangle - \langle\psi_b|Q(F_N)(\varphi-\psi_c)\rangle - \langle\psi-\psi_b|Q(F_N)\psi_c\rangle$$
$$+\sum_{n=0}^{N}\langle\psi_b|Q(E_n)\psi_c\rangle - \langle\psi_b|Q(E)\psi_c\rangle$$
$$-\langle\psi-\psi_b|Q(E)(\varphi-\psi_c)\rangle + \langle\psi_b|Q(E)(\varphi-\psi_c)\rangle + \langle\psi-\psi_b|Q(E)\psi_c\rangle\,.$$

Since $||Q(E)||\,, ||Q(F_N)|| \leq ||Q(\mathbb{R})|| = 1$, we have the estimate

$$|\Delta_N| \leq 2||\psi-\psi_b||||\varphi-\psi_c|| + 2||\psi_b||||\varphi-\psi_c|| + 2||\psi-\psi_b||||\psi_c|| + \left|\sum_{n=0}^{N}\langle\psi_b|Q(E_n)\psi_c\rangle - \langle\psi_b|Q(E)\psi_c\rangle\right|.$$

This inequality concludes the proof: given $\psi, \varphi \in \mathsf{H}_\omega$, since $\mathcal{D}_\omega$ is dense, we can fix $\psi_b, \psi_c \in \mathcal{D}_\omega$ such that the sum of the first three addends is bounded by $\epsilon/2$. Finally, exploiting the fact that $\mathscr{B}(\mathbb{R}) \ni E \mapsto \langle\psi_b|Q^{(a)}(E)\psi_c\rangle$ is $\sigma$-additive, we can fix $N$ sufficiently large that the last addend is bounded by $\epsilon/2$. So, if $\epsilon > 0$, there is $N_\epsilon$ such that $|\Delta_N| < \epsilon$ if $N > N_\epsilon$ as wanted. Notice that the series $\sum_{n=0}^{+\infty}\langle\psi|Q(E_n)\varphi\rangle$ can be re-ordered arbitrarily since we have proved that its sum is $\langle\psi|Q(E)\varphi\rangle$ which does not depend on the order used to label the sets $E_n$ because $E := \cup_{n\in\mathbb{N}} E_n$. The function $\mathscr{B}(\mathbb{R}) \ni B \mapsto \langle\psi|Q(B)\varphi\rangle \in \mathbb{C}$ is *unconditionally* $\sigma$-additive as we wanted to prove.
(b) Since the measures $\mu^{(a)}_{\omega_b}$ are solutions of the moment problem for the couples $(a, \omega^{(a)}_b)$, for $k = 0, 1, 2, \ldots$ and $\psi_b \in \mathcal{D}_\omega$, we have

$$\langle\psi_b|\pi_\omega(a)^k\psi_b\rangle = \omega_b(a^k) = \int_{\mathbb{R}} \lambda^k d\mu_{\omega_b}(\lambda) = \int_{\mathbb{R}} \lambda^k dQ^{(a,\omega)}_{\psi_b,\psi_b}(\lambda)\,. \tag{48}$$

Choosing $k = 2$ we obtain

$$||\pi_\omega(a)\psi||^2 = \int_{\mathbb{R}} \lambda^2 dQ^{(a,\omega)}_{\psi,\psi}(\lambda) \quad \text{for every } \psi \in D(\pi_\omega(a)) = \mathcal{D}_\omega,$$

which, in particular, also implies (40). It remain to be established the identity

$$\langle\varphi|\pi_\omega(a)\psi\rangle = \int_{\mathbb{R}} \lambda dQ^{(a,\omega)}_{\varphi,\psi}(\lambda) \quad \text{for every } \varphi \in \mathsf{H}_\omega \text{ and } \psi \in D(\pi_\omega(a))\,. \tag{49}$$

From (48) with $k = 1$ we conclude that, for every $\psi_b \in \mathcal{D}_\omega$,

$$\langle \psi_b | \pi_\omega(a) \psi_b \rangle = \int_{\mathbb{R}} \lambda dQ^{(a,\omega)}_{\psi_b,\psi_b}(\lambda) = \langle \psi_b | A \psi_b \rangle ,$$

where $A$ is the Hermitian operator uniquely constructed out of the POVM $Q^{(a)}$ according to Theorem 3.13. Notice that the domain of $A$ is $\left\{ \psi \in \mathsf{H}_\omega \;\middle|\; \int_{\mathbb{R}} \lambda^2 dQ^{(a,\omega)}_{\psi,\psi}(\lambda) < +\infty \right\}$ that includes $\mathcal{D}_\omega$ for (40) that we have already proved. Polarization identity applied to both sides of $\langle \psi_b | \pi_\omega(a) \psi_b \rangle = \langle \psi_b | A \psi_b \rangle$ immediately proves that $\langle \varphi | \pi_\omega(a) \psi \rangle = \langle \varphi | A \psi \rangle$ for every pair $\varphi, \psi \in \mathcal{D}_\omega$. Since this space is dense, we conclude that $\pi_\omega(a) = A\restriction_{\mathcal{D}_\omega}$. Identity (49) is therefore valid as an immediate consequence of (b) Theorem 3.13.
(c) Everything is an immediately arises from (d), (e), (f) of Theorem 3.10.
(d) (i) is true per direct inspection. (ii) is consequence of (39) using $\psi_b \in \mathsf{H}_{\omega_b} \subset \mathsf{H}_\omega$. (iii) arises from the fact that $Q^{(a,\omega)}$ decomposes $\pi_\omega(a)$, (42) and (10). □

**Remark 4.7** Item (d) physically states that the POVMs $Q^{(a,\omega_b)}$ are consistent with both the expectation-value interpretation of each $\omega_b(a)$ and the interpretation of every $\overline{\pi_{\omega_b}(a)}$ as generalized observable. We stress that, if $Q^{(a,\omega)}$ is a PVM because, for example, $\overline{\pi_\omega(a)}$ is selfadjoint, it is still possible that $Q^{(a,\omega_b)}$ is a merely a POVM and not a PVM.■

Next theorem can be considered as a *weaker* version of both Theorem 2.3 and its converse (the proper converse of Theorem 2.3 does not exist as we have seen).

**Theorem 4.8** *Let $\mathfrak{A}$ be a unital $^*$-algebra, $\omega\colon \mathfrak{A} \to \mathbb{C}$ a finite weight and $a = a^* \in \mathfrak{A}$. $\overline{\pi_\omega(a)}$ is maximally symmetric (in particular selfadjoint), if and only if there exists a unique family of consistent measures $\{\mu^{(a)}_{\omega_b}\}_{b\in\mathfrak{A}}$ solutions of the moment problem (12) relative to the pairs $(a, \omega_b)$ for $b \in \mathfrak{A}$.*

*Proof.* It immediately follows from (c) of Theorem 4.6 and (39). □

## 5 Conclusions and open problems

Let us recap the issues we tackled and summarize the results we found regarding a generic unital $^*$-algebra $\mathfrak{A}$, the GNS construction and its use in quantum theory. Some open issues are listed in the final section.

### 5.1 Summary

*Issue A* concerned the fact that an Hermitian element $a^* = a \in \mathfrak{A}$ may be represented in a GNS representation of some finite weight $\omega$ by means of an operator $\pi_\omega(a)$ which is not essentially selfadjoint and which can or cannot have selfadjoint extensions (see (1) and (2) in Example 1.3). Referring to the standard Hilbert-space formulation of quantum theory, it is not obvious in these case how we can think of $a$ as an abstract observable. (If $\mathfrak{A}$ is a $C^*$-algebra, $\pi_\omega(a)$ is always selfadjoint making Issue A harmless.)

We have seen in this work that it is always however possible to interpret the symmetric operator $\pi_\omega(a)$ as a *generalized observable* just by fixing a normalized POVM associated to it (these normalized POVMs are the same as those of $\overline{\pi_\omega(a)}$) which decompose the operator $\pi_\omega(a)$ according to Definition 3.12 into a generalized version of the spectral theorem of selfadjoint operators based on PVMs. The POVM decomposing $\pi_\omega(a)$ is unique if and only if $\overline{\pi_\omega(a)}$ is *maximally symmetric* (Definition 3.6) and this unique normalized POVM is a PVM when $\pi_\omega(a)$ is essentially selfadjoint in particular. The class of normalized POVMs associable to $\pi_\omega(a)$ include all possible PVMs arising from all possible selfadjoint extensions of $\pi_\omega(a)$ if any.

*Issue B* regarded the popular expectation-value interpretation of $\omega(a)$, i.e., the existence of a Borel measure $\mu_\omega^{(a)}$ over $\mathbb{R}$ satisfying (5). In principle $\mu_\omega^{(a)}$ can be fixed looking for a measure giving rise to the known momenta $\omega(a^n)$, that is solving the moment problem (6). If $\overline{\pi_\omega(a)}$ is selfadjoint, there is a natural physically meaningful way (7) to define $\mu_\omega^{(a)}$ using the PVM of $\overline{\pi_\omega(a)}$. It happens that, even if $\overline{\pi_\omega(a)}$ is not selfadjoint and, in particular when $\pi_\omega(a)$ does not admit selfadjoint extensions, (generally many) measures $\mu_\omega^{(a)}$ associated with the class of moments $\omega(a^n)$ as in (6) do exist. These measures do not have a direct spectral meaning as it is expected from quantum theory. In general, the complete class of these measures for a fixed $a$ and $\omega$ is very large and the physical meaning of them is dubious.

In this work, we have seen that to reduce the number of the measures $\mu_\omega^{(a)}$, it is convenient to refer to the information provided by other elements $b \in \mathfrak{A}$ in terms of perturbed weights $\omega_b$ (Definition 2.1) and to consider $\mu_\omega^{(a)}$ as an element of the larger class of measures $\mu_{\omega_b}^{(a)}$ solving separately the moment problem for $a$ and each $\omega_b$. As a matter of fact, the measures $\mu_{\omega_b}^{(a)}$ are expected to enjoy a list of physically meaningful mutual relations (33)-(34) able to considerably reduce their number. A class of those measures, for $a$ and $\omega$ fixed satisfying constraints (33)-(34) when $b \in \mathfrak{A}$ is called *consistent class of measures* (Definition 4.2). (If $\mathfrak{A}$ is a $C^*$-algebra, there is exactly one measure $\mu_\omega^{(a)}$ spectrally obtained implementing the expectation value intepretation of $\omega(a)$ and Issue B is harmless.)

A first result, Theorem 4.4 establishes that every normalized POVM decomposing the symmetric operator $\pi_\omega(a)$, seen as a generalized observable, defines a unique class of consistent measures $\{\nu_b^{(a)}\}_{b\in\mathfrak{A}}$ in the natural way (37). These measures solve the moment problem for $\omega_b$, thus corroborating the expectation-value interpretation of $\omega_b(a)$ and $\omega(a)$ in particular. When the said POVM is a PVM, the standard relation (7) between PVMs and Borel spectral measures is recovered.

The results is reversed in Theorem 4.6, which is the main achievement of this paper: for a Hermitian element $a \in \mathfrak{A}$ and a finite weight $\omega$, an associated consistent class of measures $\mu_{\omega_b}^{(a)}$ solving the moment problem for every corresponding perturbation $\omega_b$ always determines a unique POVM which decomposes the symmetric operator $\pi_\omega(a)$.

The couple of mentioned theorems proves that the family of all normalized POVMs

$Q^{(a,\omega)}$ decomposing the generalized observable $\pi_\omega(a)$, for a fixed pair $(a, \omega)$, is one-to one with the full family of consistent classes of measures $\{\mu^{(a)}_{\omega_b}\}_{b\in\mathfrak{A}}$ which implement the expectation-value interpretation of every perturbation $\omega_b(a)$ (including $\omega(a)$ for $b = \mathbb{I}$).

Actually Theorem 4.6 proves little more. Also the measures $\mu^{(a)}_{\omega_b}$ arise from POVMs $Q^{(a,\omega_b)}$ (43) which, as expected, decompose the respective generalized observables $\pi_{\omega_b}(a)$. The POVMs $Q^{(a,\omega_b)}$ are all mastered by the initial POVM $Q^{(a,\omega)}$ (42).

As a complement, Theorem 4.8 establishes that $\overline{\pi_\omega(a)}$ is maximally symmetric (selfadjoint in particular) if and only if there is only a unique class of consistent measures $\mu^{(a)}_{\omega_b}$ solving the moment problem for every corresponding perturbation $\omega_b$.

Part of Theorem 4.8 admits a stronger version established in Theorem 2.3 which refers to the whole class of measures $\mu^{(a)}_{\omega_b}$ solving the moment problem for every corresponding perturbation $\omega_b$ without imposing constrains (33)-(34). If $a = a^*$ and $\omega$ are fixed and there is exactly one measure solving the moment problem for each perturbation $\omega_b$, then $\overline{\pi_\omega(a)}$ is selfadjoint and therefore has the interpretation of a standard observable in the usual Hilbert space formulation of quantum theories. The converse assertion of this very strong result is untenable, as explicitly proved with a counter-example (Example 2.5).

### *5.2 Open issues*

There are at least two important open issues after the results established in this work. One concerns the fact that, when $\pi_\omega(a)$ is only symmetric, its intepretation as generalized observable depends on the choice of the normalized POVM associated to it. This POVM is unique if and only if $\overline{\pi_\omega(a)}$ is maximally symmetric (selfadjoint in particular). It is not clear if the information encapsulated in $\mathfrak{A}$, $a$, $\omega$ permits one to fix this choice or somehow reduce the number of possibilities. The second open issue regards the option of simultaneous measurements of compatible (i.e., pairwise commuting) abstract observables $a_1, a_n, \ldots, a_n$ with associated joint measures on $\mathbb{R}^n$ accounting for the expectation-value interpretation. The many-variables moment problem is not a straightforward generalization of the one-variable moment problem [13] and also the notion of joint POVM presents some non-trivial technical difficulties [2]. Already at the level of selfadjoint observables, commutativity of symmetric operators (say $\pi_\omega(a_1)$ and $\pi_\omega(a_2)$) on a dense invariant domain of essential selfadjointness ($\mathcal{D}_\omega$) does not imply the much more physically meaningful commutativity of their respective PVMs (as proved by Nelson [14]) and the existence of a joint PVM. These issues will be investigated elsewhere.

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# A Appendix

### A.1 Reducing subspaces

**Definition A.1** If $D(T) \subset \mathsf{H}$ is a subspace of the Hilbert space $\mathsf{H}$, let $T : D(T) \to \mathsf{H}$ be an operator and $\mathsf{H}_0 \subset \mathsf{H}$ a closed subspace with $\{0\} \neq \mathsf{H}_0 \neq \mathsf{H}$, $P_0$ denoting the orthogonal projector onto $\mathsf{H}_0$. In this case, $\mathsf{H}_0$ is said to **reduce** $T$ if both conditions are true

(i) $T(D(T) \cap \mathsf{H}_0) \subset \mathsf{H}_0$ and $T(D(T) \cap \mathsf{H}_0^\perp) \subset \mathsf{H}_0^\perp$,

(ii) $P_0(D(T)) \subset D(T)$,

so that the direct orthogonal decomposition holds

$$D(T) = (D(T) \cap \mathsf{H}_0) \oplus (D(T) \cap \mathsf{H}_0^\perp) \quad \text{and} \quad T = T\restriction_{D(T)\cap\mathsf{H}_0} \oplus T\restriction_{D(T)\cap\mathsf{H}_0^\perp} . \tag{50}$$

The operator $T = T\restriction_{D(T)\cap\mathsf{H}_0}$ is called the **part of** $T$ **on** $\mathsf{H}_0$. ■

**Remark A.2**
**(1)** It is worth stressing that (i) does *not* imply (ii) and without (ii) the direct orthogonal decomposition (50) cannot take place.
**(2)** Evidently $\mathsf{H}_0$ reduces $T$ iff $\mathsf{H}_0^\perp$ reduces $T$, since $I - P_0$ projects onto $\mathsf{H}_0^\perp$.
**(3)** A condition equivalent to (i)-(ii) is $P_0 T \subset T P_0$ as the reader immediately proves. ■

**Proposition A.3** *If the closed subspace $\mathsf{H}_0$ reduces the symmetric (selfadjoint) operator $T$ on $\mathsf{H}$, then also $T\restriction_{D(T)\cap\mathsf{H}_0}$ and $T\restriction_{D(T)\cap\mathsf{H}_0^\perp}$ are symmetric (resp. selfadjoint) in $\mathsf{H}_0$ and $\mathsf{H}_0^\perp$ respectively.*

*Proof.* Direct inspection. □

A useful technical fact is presented in the following proposition [12].

**Proposition A.4** *Let $T$ be a closed symmetric operator on a Hilbert space $\mathsf{H}$. Let $D_0$ be a dense subspace of a closed subspace $\mathsf{H}_0$ of $\mathsf{H}$ such that $D_0 \subset D(T)$ and $T(D_0) \subset \mathsf{H}_0$. Suppose that $T_0 := T\restriction_{D_0}$ is essentially self-adjoint on $\mathsf{H}_0$. Then $\mathsf{H}_0$ reduces $T$ and $\overline{T_0}$ is the part of $T$ on $\mathsf{H}_0$.*

*Proof.* See [12, Prop.1.17]. □

### A.2 More on generalized symmetric and selfadjoint extensions

According to definition 3.8 we have

$$D(A) \subset D(B) \cap \mathsf{H} \subset D(B) , \tag{51}$$

where $D(B)$ is dense in $\mathsf{K}$ and $D(A)$ is dense in $\mathsf{H}$. Generalized extensions $B$ with $B \supsetneq A$ are classified accordingly to the previous inclusions following [1], in particular:

(i) $B$ is said to be of **kind I** if $D(A) \neq D(B) \cap \mathsf{H} = D(B)$ – that is, if $B$ is a standard extension of $A$;

(ii) $B$ is said to be of **kind II** if $D(A) = D(B) \cap \mathsf{H} \neq D(B)$;

(iii) $B$ is said to be of **kind III** if $D(A) \neq D(B) \cap \mathsf{H} \neq D(B)$;

**Proposition A.5** *If $A : D(A) \to \mathsf{H}$ with $D(A) \subset \mathsf{H}$ is maximally symmetric and $B \supsetneq A$ is a generalized symmetric extension, then $B$ is of kind $II$.*

*Proof.* The kind $I$ is not possible *a priory* since $A$ does not admit proper symmetric extensions in $\mathsf{H}$. Let us assume that $B$ is either of kind $II$ or $III$ and consider the operator $P_\mathsf{H} B P_\mathsf{H}$, with its natural domain $D(P_\mathsf{H} B P_\mathsf{H}) = D(BP_\mathsf{H})$, where $P_\mathsf{H} \in \mathscr{L}(\mathsf{K})$ is the orthogonal projector onto $\mathsf{H}$. Since this is a symmetric extension of $A$ in $\mathsf{H}$ which is maximally symmetric, we have $A = P_\mathsf{H} B P_\mathsf{H}$, in particular $D(BP_\mathsf{H}) = D(A)$. As a consequence, if $x \in D(B) \cap \mathsf{H}$, then $x \in D(BP_\mathsf{H}) = D(A)$ so that $D(A) \supset D(B) \cap \mathsf{H}$ and thus $D(A) = D(B) \cap \mathsf{H}$ because the other inclusion is true from (51). We have proved that $B$ is of kind $II$. □

We have an important technical result

**Theorem A.6** *A non-selfadjoint symmetric operator $A$ always admits generalized self-adjoint extension $B$. Such an extension can be chosen of kind $II$ when $A$ is closed.*

*Proof.* See [1, Thm 1. p.127 Vol II] for the former statement. The latter statement relies on the comment under the proof of [1, Thm 1. p.127 Vol II] and it is completely proved in [9, Thm13][7]. The fact that the selfadjoint extensions can be chosen in order to satisfy (iii) of Definition 3.8 is proved in [10, Thm7]. □

### *A.3 Proof of some propositions*

**Proof of Proposition 3.9**. Let $B$ be a generalized symmetric extension in the Hilbert space $\mathsf{K}$ of the selfadjoint operator $A$ in the Hilbert space $\mathsf{H}$ with $\mathsf{H} \subset \mathsf{K}$. The closed symmetric operator $B' = \overline{B}$ in $\mathsf{K}$ extends $A$. In view of Proposition A.4, since $B'\restriction_{D(A)\cap\mathsf{H}} = A$ is selfadjoint on $\mathsf{H}$, then $\mathsf{H}$ reduces $B'$ and $\overline{B} = \overline{B'\restriction_{D(A)\cap\mathsf{H}}} \oplus B''$, where the two addends are symmetric operators respectively on $\mathsf{H}$ and $\mathsf{H}^\perp$. Requirement (iii) in Definition 3.8 imposes that $\mathsf{H}^\perp = \{0\}$, so that $\mathsf{K} = \mathsf{H}$ and $B$ is a standard symmetric extension of the selfadjoint operator $A$ which entails $B = A$. □

**Proof of Theorem 3.10**. (a) and (b). If $A$ is symmetric, for each generalized selfadjoint extension $B$ as in Definition 3.8 (they exist in view of Theorem A.6 and, if $A$ is selfadjoint,

[7] Unfortunatley the necessary closedness requirement disappeared passing from [9, Thm13] to the comment under [1, Thm 1. p.127 Vol II].

$B := A$), one can define a corresponding normalized POVM $Q^{(A)} \colon \mathscr{B}(\mathbb{R}) \to \mathfrak{B}(\mathsf{H})$ as follows. Let $P \colon \mathscr{B}(\mathbb{R}) \to \mathscr{L}(\mathsf{K})$ be the unique PVM associated with $B$

$$B := \int_{\mathbb{R}} \lambda dP(\lambda)\,, \tag{52}$$

(see, e.g. [7, Thm. 9.13]) and let $P_{\mathsf{H}} \in \mathscr{L}(\mathsf{K})$ denotes the orthogonal projection onto $\mathsf{H}$ viewed as a closed subspace of $\mathsf{K}$. A normalized POVM $Q^{(A)} \colon \mathscr{B}(\mathbb{R}) \to \mathfrak{B}(\mathsf{H})$ is then defined by setting

$$Q^{(A)}(E) := P_{\mathsf{H}} P(E)\restriction_{\mathsf{H}} \qquad \forall E \in \mathscr{B}(\mathbb{R})\,. \tag{53}$$

The POVM $Q^{(A)}$ is linked to $A$ by the following identities as the reader easily proves form standard spectral theory:

$$\langle \psi | A\varphi \rangle = \int_{\mathbb{R}} \lambda dQ^{(A)}_{\psi,\varphi}(\lambda)\,, \qquad \|A\varphi\|^2 = \int_{\mathbb{R}} \lambda^2 dQ^{(A)}_{\varphi,\varphi}(\lambda)\,, \qquad \forall \psi \in \mathsf{H}, \varphi \in D(A)\,, \tag{54}$$

The above relation implies the following facts, where $\lambda^k$ henceforth denotes the map $\mathbb{R} \ni \lambda \to \lambda^k \in \mathbb{R}$ for $k \in \mathbb{N} := \{0, 1, 2, \ldots\}$,

$$\begin{aligned} A = B\restriction_{D(A)\cap \mathsf{H}}\,, \quad D(A) \subset \{\psi \in \mathsf{H} |\ \lambda \in L^2(\mathbb{R}, Q^{(A)}_{\psi,\psi})\} &= \{\psi \in \mathsf{H} |\ \lambda \in L^2(\mathbb{R}, P_{\psi,\psi})\} \\ &= D(B) \cap \mathsf{H} \subset D(B)\,, \end{aligned} \tag{55}$$

the second equality arising from the identity

$$\int_{\mathbb{R}} \lambda^2 dP_{\psi,\psi}(\lambda) = \int_{\mathbb{R}} \lambda^2 dQ^{(A)}_{\psi,\psi}(\lambda) \qquad \forall \psi \in \mathsf{H}\,,$$

which descends from equation (53). [1, Thm. 2, p. 129 Vol II] Proves that every POVM (it is equivalent to a *spectral function* used therein, see Remark 3.1) satisfying (54) arises from a PVM of a generalized selfadjoint extension $B$ of $A$ as above, so that (55) are satisfied. The proof of (a) and (b) is over.
(c) If the selfadjoint operator $B$ extending $A$ as in (b) is a standard extension of $A$, then $\mathsf{H} = \mathsf{K}$ so that $P_{\mathsf{K}} = I$ and $Q^{(A)} = P$ so that $Q^{(A)}$ is a PVM. If $Q^{(A)}$ is a PVM, $B := \int_{\mathbb{R}} \lambda dQ^{(A)}(\lambda)$ with domain $D(B) = \{\psi \in \mathsf{H} \mid \int_{\mathbb{R}} \lambda^2 dQ^{(A)}_{\psi,\psi}(\lambda) < +\infty\}$ is a standard selfadjoint extension of $A$ and the associated trivial dilation triple $\mathsf{K} := \mathsf{H}$, $P_{\mathsf{H}} := I$, $P := Q^{(A)}$ trivially generates $Q^{(A)}$ as in (b).
(d) Referring to the proof of (a) and (b) above, for generalized selfadjoint extensions $B$ of kind $II$, and this choice for $B$ is always feasible when $A$ is closed in view of Theorem A.6, we have $D(A) = D(B) \cap \mathsf{H}$ and therefore

$$\begin{aligned} A = B\restriction_{D(B)\cap \mathsf{H}}\,, \quad D(A) = \{\psi \in \mathsf{H} |\ \lambda \in L^2(\mathbb{R}, Q^{(A)}_{\psi,\psi})\} &= \{\psi \in \mathsf{H} |\ \lambda \in L^2(\mathbb{R}, P_{\psi,\psi})\} \\ &= D(B) \cap \mathsf{H}\,. \end{aligned} \tag{56}$$

Since (55) is valid for every POVM satisfying (54), we have that the identity $D(A) = \{\psi \in \mathsf{H} |\ \lambda \in L^2(\mathbb{R}, P_{\psi,\psi})\}$ holds if and only if $A = B\restriction_{D(B)\cap\mathsf{H}}$, namely $B$ is an extension of type $II$ of $A$.
(e) and (f). They are established in [1, Thm. 2, p. 135 Vol II], taking Theorem A.6 into account and (d) above. □

**Proof of Theorem 3.13**. Consider Naimark's dilation triple of $Q$, $(\mathsf{K}, P_{\mathsf{H}}, P)$ and define the selfadjoint operator $B := \int_{\mathbb{R}} \lambda dP(\lambda)$. By hypothesis $D(A^{(Q)}) = D(B)\cap\mathsf{H}$. Since $D(B)$ is a subspace of $\mathsf{K}$, it also holds that $D(A^{(Q)})$ is a subspace of $\mathsf{H}$ proving (a). A well-known counterexample due to Naimark [1] proves that, in some cases, $D(A^{(Q)}) = \{0\}$ though $Q$ is not trivial. It is clear that, if an operator $A^{(Q)} : D(A^{(Q)}) \to \mathsf{H}$ satisfies (24) then it is unique due to the arbitrariness of $\varphi \in \mathsf{H}$. So we prove (b) and (d) simultaneously just checking that $P_{\mathsf{H}} \int_{\mathbb{R}} \lambda dP(\lambda)\restriction_{D(A^{(Q)})}$ satisfies (24). The proof is trivial since, from standard properties of the integral of a PVM (see, e.g., [7]), if $\psi \in D(A^{(Q)}) \subset D(B)$ and $\varphi \in \mathsf{H}$, then

$$\left\langle \varphi \middle| P_{\mathsf{H}} \int_{\mathbb{R}} \lambda dP(\lambda)\psi \right\rangle = \left\langle P_{\mathsf{H}}\varphi \middle| \int_{\mathbb{R}} \lambda dP(\lambda)\psi \right\rangle = \int_{\mathbb{R}} \lambda dP_{P_{\mathsf{H}}\varphi,\psi}(\lambda) = \int_{\mathbb{R}} \lambda dQ_{\varphi,\psi}(\lambda)\,.$$

(c) immediately arises with the same argument taking advantage of the fact that $B = B^*$ and $P_{\mathsf{H}}\psi = \psi$ if $\psi \in \mathsf{H}$. Regarding (e), it is sufficient observing that (see, e.g. [7]), if $\psi \in D(B)$, then $||B\psi||^2 = \int_{\mathbb{R}} \lambda^2 dP_{\psi,\psi}(\lambda)$ and next taking advantage of $D(A^{(Q)}) = D(B) \cap \mathsf{H}$ and (d) observing in particular that $P^*_{\mathsf{H}} P_{\mathsf{H}}\varphi = P_{\mathsf{H}} P_{\mathsf{H}}\varphi = P_{\mathsf{H}}\varphi = \varphi$ if $\varphi \in \mathsf{H}$. The fact that $A^{(Q)}$ is closed immediately follows from the fact that $B$ is closed (because selfadjoint) and $A = B|_{D(B)\cap\mathsf{H}}$ where $\mathsf{H}$ is closed. □

**Proof of Proposition 4.5**. What we have to prove is just that the right-hand side of (38) is a positive semi definite Hermitian scalar product over $X$. Indeed, with that definition of the scalar product $(\,|\,)_p$, the identity $p(x) = \sqrt{(x|x)_p}$ is valid (see below) and this fact automatically implies that $p$ is a seminorm. Uniqueness of the scalar product generating a seminorm $p$ is a trivial consequence of the polarization identity. Let us prove that $(\,|\,)_p$ defined in (38),

$$(x|y)_p := \frac{1}{4}\sum_{k=0}^{3}(-i)^k p(x+i^k y)^2 \qquad \text{for } x, y \in X,$$

is a positive semidefinite Hermitian scalar product over $X$. From the definition of $(\,|\,)_p$ and property (i) of $p : X :\to [0, +\infty)$, it is trivial business to prove the following facts per direct inspection:

1. $(x|x)_p = p(x)^2 \geq 0$,

2. $(x|0)_p = 0$,

3. $(x|y)_p = \overline{(y|x)_p}$,

4. $(x|iy)_p = i(x|y)_p$.

With these identities, we can also prove

5. $(x|y+z)_p = (x|y)_p + (x|z)_p$,

6. $(x|-y)_p = -(x|y)_p$,

by exploiting property (ii) of $p$. Actually, (6) immediately arises form (2) and (5). We will prove (5) as the last step of this proof. Iterating property (5), we easily obtain $(x|ny)_p = n(x|y)_p$ for every $n \in \mathbb{N}$, so that $(1/n)(x|z)_p = (x|(1/n)z)_p$ when replacing $ny$ for $z$. As a consequence, $\lambda(x|y)_p = (x|\lambda y)_p$ if $\lambda \in \mathbb{Q}$. This results extends to $\lambda \in \mathbb{R}$ if the map $\mathbb{R} \ni \lambda \mapsto (x|\lambda y)_p$ is right-continuous because, for every $x \in [0, +\infty)$ there is a decreasing sequence of rationals tending to it. The definition (38) of $(x|y)_p$ proves that map is in fact right-continuous since property (iii) of $p$ implies that, if $\lambda_0 \in [0, +\infty)$,

$$p(x + \lambda i^k y) = p((x + \lambda_0 i^k y) + (\lambda - \lambda_0) i^k y) \to p(x + \lambda_0 i^k y) \quad \text{for } \mathbb{R} \ni \lambda \to \lambda_0^+ \,.$$

We can therefore add the further property

7. $\lambda(x|y)_p = (x|\lambda y)_p$ if $\lambda \in [0, +\infty)$.

Collecting properties (1), (3), (5), (7), (6), (4) together, we obtain that $X \times X \ni (x, y) \mapsto (x|y)_p$ defined as in (38) is a a positive semi definite Hermitian scalar product over $X$ whose associated seminorm is $p$ as wanted.
To conclude the proof, we establish property (5) from requirement (ii) on $p$.

$$4(x|y+z)_p = \sum_{k=0}^{3} (-i)^k p(x + i^k(y+z))^2 = \sum_{k=0}^{3} (-i)^k p((x/2 + i^k y) + (x/2 + i^k z))^2 \,.$$

Since, from (i) of the requirements on $p$,

$$\sum_{k=0}^{3} (-i)^k p((x/2 + i^k y) - (x/2 + i^k z))^2 = \sum_{k=0}^{3} (-i)^k p(i^k(y-z))^2 = \sum_{k=0}^{3} (-i)^k p((y-z))^2 = 0,$$

we can re-arrange the found decomposition of $4(x|y+z)_p$ as

$$4(x|y+z)_p = \sum_{k=0}^{3} (-i)^k \left[ p((x/2 + i^k y) + (x/2 + i^k z))^2 - p((x/2 + i^k y) - (x/2 + i^k z))^2 \right] \,.$$

From (ii) of the requirements on $p$,

$$4(x|y+z)_p = \sum_{k=0}^{3} (-i)^k \left[ 2p(x/2 + i^k y))^2 + 2p(x/2 + i^k z))^2 \right] = 8(x/2|y)_p + 8(x/2|z)_p \,.$$

The special case $z = 0$ and (2) yield $2(x/2|y)_p = (x|y)_p$ which, exploited again above yields wanted result (5), $4(x|y+z)_p = 4(x|y)_p + 4(x|z)_p$. □

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