

Massive vector bosons: is the geometrical interpretation as a spontaneously broken gauge theory possible at all scales?

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Abstract

The usual derivation of the Lagrangian of a model for massive vector bosons, by spontaneous symmetry breaking of a gauge theory, implies that the prefactors of the various interaction terms are uniquely determined functions of the coupling constant(s) and the masses. Since, under the renormalization group (RG) flow, different interaction terms get different loop-corrections, it is uncertain, whether these functions remain fixed under this flow. We investigate this question for the $U(1)$ -Higgs-model to 1-loop order in the framework of Epstein-Glaser renormalization. Our main result reads: choosing the renormalization mass scale(s) in a way corresponding to the minimal subtraction scheme, the geometrical interpretation as a spontaneously broken gauge theory gets lost under the RG-flow. This holds also for the clearly stronger property of BRST-invariance of the Lagrangian. On the other hand we prove that physical consistency, which is a weak form of BRST-invariance of the time-ordered products, is maintained under the RG-flow.

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1 Introduction

The classical geometrical concepts of fibre bundles and group theory have been crucial for the development of quantum gauge theories, and the Higgs mechanism was the key to incorporate the electroweak interaction into the framework of renormalizable field theory.

However, to the best of our knowledge, the Higgs mechanism is not well understood on a purely quantum level. And it is not needed: starting with massive BRST-invariant free fields, making a general renormalizable ansatz for the interaction and requiring physical consistency (PC) [KO79, DS00, Gri00] or perturbative gauge invariance (PGI) [DS99, ADS99, Sch01, DGBSV10, Sch10]¹ one obtains a consistent perturbative quantum theory of massive vector bosons. (Some obvious properties as Poincaré invariance and relevant discrete symmetries are also taken into account.) PC is the condition that the free BRST-charge² commutes with the “ S -matrix” in the adiabatic limit, in order that the latter induces a well-defined operator on the physical subspace; PGI is a refinement of this condition which is formulated independently of the adiabatic limit – a sufficient (but in general not necessary) condition for PC. If the ansatz for the interaction contains only trilinear and quadrilinear fields³ the resulting Lagrangian is essentially unique and agrees with what one obtains from spontaneous symmetry breaking of a gauge theory; in particular the presence of Higgs particles and chirality of fermionic interactions can be understood in this way without recourse to any geometrical or group theoretical concepts [Sto97, DS99]. These derivations of the interaction from basic QFT-principles use PGI (or PC) only on the level of tree diagrams (PGI-tree).

¹PGI was first introduced in [DHKS94]; in [DGBSV10] it is called ‘causal gauge invariance’.

²That is the charge implementing the BRST-transformation of the asymptotic free fields.

³Throughout this paper we use the words bilinear, trilinear and quadrilinear in the sense of bi-, tri- and quadrilinear in the basic fields.

In the literature the geometrical interpretation of the Standard Model of electroweak interactions as a spontaneously broken gauge theory is frequently used at several (or even all) scales. This is evident for the cosmological models relying on the “electroweak phase transition”. Or, looking carefully at the geometrical derivations of a value for the Higgs mass of Connes et al. ([CCM07] and references cited therein) and Tolksdorf [TT07], we realized that in these papers the geometrical interpretation is used at two very distinct scales: at the Z -mass and at the unification scale.

This paper was initiated by serious doubts about the geometrical interpretability at all scales, which rely on the following: this interpretation is possible iff the prefactors of the various interaction terms (i.e. of the vertices) are prescribed functions of the coupling constant(s) and masses. Since different vertices get different loop-corrections it is uncertain, whether these functions remain fixed under the RG-flow.

Similarly to the conventional literature [Sib], our RG-flow depends strongly on the renormalization scheme. Naively one might think that this is not so, because we define the RG-flow by a scaling transformation [HW03, DF04, BDF09].⁴ But the scheme dependence comes in by the choice of the renormalization mass scale(s) M : the scaling transformation may act on M or it may not, and different choices for different Feynman diagrams are possible.

An important result of this paper is that *PC is maintained under the renormalization group (RG) flow* (Sect. 5). It is well known that also renormalizability (by power counting) is preserved. But, our original hope that these two properties yield enough information about the running interaction to answer the geometrical interpretability, turned out to be too optimistic. Due to the presence of bilinear fields, PC and renormalizability are much less restrictive than in the above mentioned calculations involving only tri- and quadrilinear fields.

For this reason we proceed in a less elegant way: we answer the geometrical interpretability by means of a lot of explicit 1-loop computations of the RG-flow (Sect. 6). Since, up to a few scalar field examples in [DF04, BDF09], such calculations have not yet been done in the framework of Epstein-Glaser renormalization, we explain them in detail (see Sects. 6.1-6.2 and Appendices A-B).

To get information about the important question whether PGI is maintained under the RG-flow, we analyze PGI-tree for the running interaction (Sec. 7).

BRST-invariance of the Lagrangian is a property which is truly stronger than the geometrical interpretability and also stronger than PGI-tree. We investigate whether it can be preserved under the RG-flow by a suitable renormalization prescription (Sects. 3 and 6).

We assume that the reader is familiar with the formalism for Epstein-Glaser renormalization (also called “causal perturbation theory”) given in [DF04], in particular we will use the Main Theorem, which is the basis for our definition of the RG-flow, and the

⁴This seems to be the obvious way to introduce the RG-flow in the Epstein-Glaser framework [EG73]. Namely, in this framework renormalization is the extension of distributions (see footnote 8) and, as long as the adiabatic limit (17) is not performed, renormalization in this sense cannot be interpreted as a redefinition of fields, masses and coupling constants depending on a mass scale.

scaling and mass expansion [HW02, Düt15].

2 Precise formulation of the question

The Lagrangian of the model: to simplify the calculations we study only one massive vector field A^μ , the corresponding Stückelberg field B , a further real scalar field φ (usually called ‘‘Higgs field’’) and the Fadeev-Popov ghost fields (u, \tilde{u}) . We work with the free Lagrangian

$$\begin{aligned} L_0^{\mathbf{m},\Lambda} = & -\frac{1}{4}F^2 + \frac{m^2}{2}(A \cdot A) + \frac{1}{2}(\partial B \cdot \partial B) - \frac{m_B^2}{2}B^2 - \frac{\Lambda}{2}(\partial A)^2 \\ & + \frac{1}{2}(\partial\varphi \cdot \partial\varphi) - \frac{m_H^2}{2}\varphi^2 + \partial\tilde{u} \cdot \partial u - m_u^2 \tilde{u}u, \end{aligned} \quad (1)$$

where $F^2 := (\partial^\mu A^\nu - \partial^\nu A^\mu)(\partial_\mu A_\nu - \partial_\nu A_\mu)$, $\mathbf{m} := (m, m_B, m_u, m_H)$ denotes the masses of the various basic fields and Λ is the gauge-fixing parameter.

For the moment we do not care about any notion of gauge symmetry and admit interactions of the form

$$\begin{aligned} L_{\kappa,\lambda}^{\mathbf{m},\Lambda} = & \kappa \left(m(A \cdot A)\varphi - \frac{\lambda_{10} m_u^2}{m} \tilde{u}u\varphi + \lambda_1 B(A \cdot \partial\varphi) \right. \\ & \left. - \lambda_2 \varphi(A \cdot \partial B) - \frac{\lambda_3 m_H^2}{2m} \varphi^3 - \frac{\lambda_4 m_H^2}{2m} B^2 \varphi \right) \\ & + \kappa^2 \left(\frac{\lambda_5}{2} (A \cdot A)\varphi^2 + \frac{\lambda_6}{2} (A \cdot A)B^2 - \frac{\lambda_7 m_H^2}{8m^2} \varphi^4 \right. \\ & \left. - \frac{\lambda_8 m_H^2}{4m^2} \varphi^2 B^2 - \frac{\lambda_9 m_H^2}{8m^2} B^4 + \lambda_{11} (A \cdot A)^2 \right) \\ & + ((\lambda_{12} - 1)m + \sqrt{\Lambda} m_B) (A \cdot \partial B), \end{aligned} \quad (2)$$

where κ is the coupling constant and $\lambda := (\lambda_1, \dots, \lambda_{12})$ are arbitrary real parameters. Apart from the last, bilinear term, each field monomial in $L_{\kappa,\lambda}^{\mathbf{m},\Lambda}$ has its own, independent coupling constant $\kappa\lambda_j$ or $\kappa^2\lambda_j$. The reason for the complicated definition of λ_{12} will become clear below in (6)-(7). The free Lagrangian is parametrized by \mathbf{m} and Λ ; the interaction L has 13 additional parameters: κ and the dimensionless coupling parameters λ . We point out that at the present stage we do **not** assume the usual mass relations $m_B = m_u = \frac{m}{\sqrt{\Lambda}}$, we consider m , m_B and m_u as independent parameters.

The set of monomials appearing in $L_{\kappa,\lambda}^{\mathbf{m},\Lambda}$ (2) is the minimal set with the following properties:

- $(L_0^{\mathbf{m},\Lambda} + L_{\kappa,\lambda}^{\mathbf{m},\Lambda})$ contains all monomials which appear in the Lagrangian of the $U(1)$ -Higgs model;
- computing the RG-flow for the model given by $(L_0^{\mathbf{m},\Lambda} + L_{\kappa,\lambda}^{\mathbf{m},\Lambda})$, there do not appear any new field monomials in the running interaction, except for a constant field $k \in \mathbb{C}$ (see (19)), i.e. the set of field monomials appearing in $(L_0^{\mathbf{m},\Lambda} + L_{\kappa,\lambda}^{\mathbf{m},\Lambda})$ is stable under the RG-flow.

We point out that each term in L_0 and L is even under the field parity transformation

$$(A, B, \varphi, u, \tilde{u}) \mapsto (-A, -B, \varphi, u, \tilde{u}) . \quad (3)$$

Setting $\tilde{u} := 0$ and $u := 0$ and ignoring the $A\partial B$ -term, the set of monomials appearing in (2) can be characterized as follows: apart from $B\varphi\partial A = \partial(B\varphi A) - BA\partial\varphi - \varphi A\partial B$, these are all trilinear and quadrilinear field monomials which are Lorentz invariant, have mass dimension ≤ 4 and respect the symmetry (3).

Geometrical interpretation: by the ‘‘classical version’’ of the model $L_0 + L$ we mean $L_0 + L - L_{\text{gf}} - L_{\text{ghost}}$, where

$$L_{\text{gf}}^{m_B, \Lambda} := -\frac{\Lambda}{2} \left(\partial \cdot A + \frac{m_B}{\sqrt{\Lambda}} B \right)^2 \quad (4)$$

is the gauge-fixing term and

$$L_{\text{ghost}}^{m_u, \kappa\lambda_{10}} := \partial\tilde{u} \cdot \partial u - m_u^2 \tilde{u}u - \frac{\kappa\lambda_{10} m_u^2}{m} \tilde{u}u\varphi \quad (5)$$

is the ghost term, which is the sum of all terms in $L_0 + L$ containing the ghost fields \tilde{u}, u .

There is a distinguished choice of the parameters λ : by straightforward calculation we find that the classical version of $L_0 + L$ can be geometrically interpreted as a spontaneously broken $U(1)$ -gauge model iff the parameters λ have the values

$$\lambda_1 = \dots = \lambda_9 = 1 , \quad \lambda_{11} = \lambda_{12} = 0 . \quad (6)$$

Explicitly, these values of the parameters are equivalent to

$$L_0 + L - L_{\text{gf}} - L_{\text{ghost}} - \sqrt{\Lambda} m_B \partial_\mu (A^\mu B) = -\frac{1}{4} F^2 + \frac{1}{2} (D^\mu \Phi)^* D_\mu \Phi - V(\Phi) , \quad (7)$$

where

$$\Phi := iB + \frac{m}{\kappa} + \varphi , \quad D^\mu := \partial^\mu - i\kappa A^\mu \quad (8)$$

and

$$V(\Phi) := \frac{\kappa^2 m_H^2}{8m^2} (\Phi^* \Phi)^2 - \frac{m_H^2}{4} (\Phi^* \Phi) + \frac{m_H^2 m^2}{8\kappa^2} . \quad (9)$$

The minima of the potential $V(\Phi)$ are on the circle $\Phi = \frac{m}{\kappa} e^{i\alpha}$, $\alpha \in [0, 2\pi)$. The choice of a minima, usually one takes $\Phi_{\text{min}} = \frac{m}{\kappa}$, breaks the $U(1)$ symmetry ‘spontaneously’ and the fields φ and B are the deviations from Φ_{min} in radial and tangential direction.

Besides m, m_H, κ and Λ , also the parameters m_B, m_u and λ_{10} are not restricted by the geometrical interpretation (7). The latter are usually fixed as follows:

- the bilinear mixed term $\sim A\partial B$ in $L_0 + L$ hampers the particle interpretation. For $\lambda_{12} = 0$ (as required by the geometrical interpretation (6)), the condition that the $A\partial B$ -term vanishes is equivalent to the mass relation

$$m_B = \frac{m}{\sqrt{\Lambda}} . \quad (10)$$

- In the next section we will see that *BRST-invariance of the total Lagrangian* $L_0 + L$ implies the geometrical interpretation (7), however it restricts also the ghost parameters. Explicitly, BRST-invariance of the total Lagrangian is equivalent to the parameter values (6) and

$$m_u^2 = \frac{m_B m}{\sqrt{\Lambda}} \quad \text{and} \quad \lambda_{10} = 1 ; \quad (11)$$

note that this holds also in the presence of an $A\partial B$ -term, i.e. the validity of (10) is not assumed here.

The main aim of this paper is the following: we will start with the $U(1)$ -Higgs model, i.e. with the parameter values (6), (10) and (11), and with that we will investigate *whether the parameter values (6) are stable under the RG-flow generated by scaling transformations*, i.e. we study the question *whether the geometrical interpretation (7) is possible 'at all scales'*.

Definition of the RG-flow: from now on we will use the just mentioned initial values (6), (10) and (11). With that we have only two independent masses $\mathbf{m} := (m, m_H)$, and the interaction $L \equiv L^{\mathbf{m}} \equiv L_{\kappa}^{\mathbf{m}, \Lambda}$ is of the form

$$L = \kappa L_1 + \kappa^2 L_2 . \quad (12)$$

In view of Epstein-Glaser renormalization [EG73], we introduce an adiabatic switching of the coupling constant by a test function $g \in \mathcal{D}(\mathbb{R}^4)$:

$$L(g) \equiv L^{\mathbf{m}}(g) := \int dx \left(\kappa g(x) L_1(x) + (\kappa g(x))^2 L_2(x) \right) . \quad (13)$$

Following [HW03, DF04, BDF09] we define the RG-flow by means of a scaling transformation of the fields

$$\sigma_{\rho}^{-1}(\phi(x)) = \rho \phi(\rho x) , \quad \phi = A^{\mu}, B, \varphi, u, \tilde{u} , \quad \rho > 0 , \quad (14)$$

and a simultaneous scaling of the masses $\mathbf{m} \mapsto \rho^{-1}\mathbf{m} = (\rho^{-1}m, \rho^{-1}m_H)$; see [DF04] for the precise definition of σ_{ρ} . Under this transformation the classical action is invariant (up to a scaling of the switching function g); namely, due to $\sigma_{\rho}^{-1} L^{\rho\mathbf{m}}(x) = \rho^4 L^{\mathbf{m}}(\rho x)$ and the same for L_0 , we have

$$\int dx L_0^{\mathbf{m}}(x) + L^{\mathbf{m}}(g) = \sigma_{\rho}^{-1} \left(\int dx L_0^{\rho\mathbf{m}}(x) + L^{\rho\mathbf{m}}(g_{\rho}) \right) , \quad g_{\rho}(x) := g(\rho x) , \quad (15)$$

where the parameters Λ, κ are suppressed since they are not affected by the scaling transformation.

In QFT scaling invariance is broken by quantum effects. To explain this more in detail, we introduce the generating functional $S(iL(g))$ of the time ordered products of $L(g)$, i.e.

$$T_n(L(g)^{\otimes n}) = \frac{d^n}{i^n d\eta^n} \Big|_{\eta=0} S(i\eta L(g)) \quad \text{or more generally} \quad T_n = S^{(n)}(0) , \quad (16)$$

which we construct inductively by Epstein-Glaser renormalization [EG73]. We use that, for a purely massive model and with a suitable (re)normalization of $S(iL(g))$, the adiabatic limit

$$\mathbf{S}[L] := \lim_{\varepsilon \downarrow 0} S(iL(g_\varepsilon)) , \quad g_\varepsilon(x) := g(\varepsilon x) , \quad (17)$$

exists, where $g(0) = 1$ is assumed [EG73, EG76].⁵ Now, computing $\mathbf{S}[L]$ for the scaled fields $\sigma_\rho^{-1}(\phi(x))$ (14) and transforming the result back by σ_ρ , we obtain a result which differs in general from $\mathbf{S}[L]$ by a change of the renormalization prescription. The Main Theorem of perturbative renormalization [PS82, DF04, HW03] implies that the transformation $\mathbf{S}_\mathbf{m}[L^\mathbf{m}] \mapsto \sigma_\rho(\mathbf{S}_{\rho^{-1}\mathbf{m}}[\sigma_\rho^{-1}(L^\mathbf{m})])$ can equivalently be expressed by a renormalization of the interaction $L^\mathbf{m} \mapsto z_\rho(L^\mathbf{m})$, explicitly

$$\sigma_\rho(\mathbf{S}_{\rho^{-1}\mathbf{m}}[\sigma_\rho^{-1}(L^\mathbf{m})]) = \mathbf{S}_\mathbf{m}[z_\rho(L^\mathbf{m})] , \quad (18)$$

where the lower index \mathbf{m} of $\mathbf{S}_\mathbf{m}$ denotes the masses of the Feynman propagators. This is explained more in detail in sect. 5.

The form of the running interaction: in Sect. 5 we will see that, with a slight restriction on the (re)normalization of $S(iL(g))$, the new interaction $z_\rho(L)$ has the form

$$\begin{aligned} z_\rho(L_\kappa^{\mathbf{m},\Lambda}) \simeq \hbar^{-1} & \left[k_\rho - \frac{1}{4} a_{0\rho} F^2 + \frac{m^2}{2} a_{1\rho} (A \cdot A) - \frac{a_{2\rho}}{2} (\partial A \cdot \partial A) \right. \\ & + \frac{1}{2} b_{0\rho} (\partial B \cdot \partial B) - \frac{m^2}{2\Lambda} b_{1\rho} B^2 + \frac{1}{2} c_{0\rho} (\partial\varphi \cdot \partial\varphi) - \frac{m_H^2}{2} c_{1\rho} \varphi^2 \\ & - \frac{m^2}{\Lambda} c_{2\rho} \tilde{u}u + b_{2\rho} m (A \cdot \partial B) \\ & + \kappa \left((1 + l_{0\rho}) m (A \cdot A) \varphi - \frac{m}{\Lambda} \tilde{u}u \varphi + (1 + l_{1\rho}) B (A \cdot \partial\varphi) \right. \\ & \left. - (1 + l_{2\rho}) \varphi (A \cdot \partial B) - \frac{(1 + l_{3\rho}) m_H^2}{2m} \varphi^3 - \frac{(1 + l_{4\rho}) m_H^2}{2m} B^2 \varphi \right) \\ & + \kappa^2 \left(\frac{(1 + l_{5\rho})}{2} (A \cdot A) \varphi^2 + \frac{(1 + l_{6\rho})}{2} (A \cdot A) B^2 - \frac{(1 + l_{7\rho}) m_H^2}{8m^2} \varphi^4 \right. \\ & \left. - \frac{(1 + l_{8\rho}) m_H^2}{4m^2} \varphi^2 B^2 - \frac{(1 + l_{9\rho}) m_H^2}{8m^2} B^4 + l_{11\rho} (A \cdot A)^2 \right] , \quad (19) \end{aligned}$$

where $k_\rho \in \hbar \mathbb{C}[[\hbar]]$ is a constant field (it is the contribution of the vacuum diagrams) and \simeq means 'equal up to the addition of terms of type $\partial^a A$ ', where $|a| \geq 1$ and A is a local field polynomial; such a $\partial^a A$ -term vanishes in the adiabatic limit. It is a peculiarity of this model that a term $\sim \partial \tilde{u} \partial u$ does not appear in $z_\rho(L)$ (if not added "by hand" – see Remark 6.3) and that there are also no trilinear and quadrilinear terms in $(z_\rho(L) - L)$ containing $\tilde{u}u$.

⁵In this paper we treat the adiabatic limit on a heuristic level, for a rigorous treatment we refer to the mentioned papers of Epstein and Glaser, in which it is shown that for purely massive models the adiabatic limit (17) exists (in the strong operator sense) and is unique (i.e. independent of the choice of g).

The dimensionless, ρ -dependent coefficients k_ρ , $a_{j\rho}$, $b_{j\rho}$, $c_{j\rho}$ and $l_{j\rho}$ will collectively be denoted by e_ρ . In principle these coefficients are computable – at least to lowest orders (see the 1-loop computations in Sects. 6.1-6.2 and Appendices A-B); however, at the present stage they are unknown. The e_ρ 's are of order $\mathcal{O}(\hbar)$ (i.e. they are loop corrections), more precisely they are formal power series in $\kappa^2\hbar$ with vanishing term of zeroth order,

$$e_\rho = \sum_{n=1}^{\infty} e_\rho^{(n)} (\kappa^2\hbar)^n, \quad e = k, a_j, b_j, c_j, l_j. \quad (20)$$

Due to $z_{\rho=1}(L) = L/\hbar$, all functions $\rho \mapsto e_\rho$ have the initial value 0 at $\rho = 1$.

Proof of (20): That e_ρ is a formal power series of the form (20) can be seen as follows. To every e_ρ there corresponds a class of Feynman diagrams with external legs according to (19). For example, the diagrams contributing to $b_{0\rho}$ have 2 external legs, both are B -fields with 0 or 1 partial derivative. The vertices are given by L (2), i.e. we have trilinear vertices $\sim \kappa$ and quadrilinear vertices $\sim \kappa^2$. For each vertex there is a factor \hbar^{-1} and for each inner line a factor \hbar . A diagram with r trilinear vertices, s quadrilinear vertices, p inner and q external lines satisfies

$$3r + 4s - 2p = q \quad (21)$$

and, hence, its contribution to $z_\rho(L)$ (19) is

$$\sim \kappa^{r+2s} \hbar^{r/2+s-q/2}. \quad (22)$$

If q is odd, $q = 2q_0 + 1$, also r is odd, $r = 2r_0 + 1$ ($q_0, r_0 \in \mathbb{N}_0$), and with that we obtain the factor

$$\kappa \hbar^{-q_0} (\kappa^2\hbar)^{r_0+s}. \quad (23)$$

If q is even, $q = 2q_0$, also r is even, $r = 2r_0$ ($q_0, r_0 \in \mathbb{N}_0$), and with that we get

$$\hbar^{-q_0} (\kappa^2\hbar)^{r_0+s}. \quad (24)$$

The contributing diagrams satisfy $n := r_0 + s \geq 1$ for $q = 2, 3$ and $n := r_0 + s - 1 \geq 1$ for $q = 4$. With that we obtain (20) – the additional factors \hbar^{-1} (for $q = 2$), $\hbar^{-1}\kappa$ (for $q = 3$) and $\hbar^{-1}\kappa^2$ (for $q = 4$) in (23)-(24) agree precisely with the prefactors in $z_\rho(L)$ (19). \square

Renormalization of the wave functions, masses, gauge-fixing parameter and the coupling constants: except for the $A\partial B$ -term, all bilinear terms of $z_\rho(L)$ do not appear in L . However, introducing new fields, which are of the form

$$\phi_\rho(x) = f_\phi(\rho) \phi(x), \quad \phi = A, B, \varphi, \quad (25)$$

where $f_\phi : (0, \infty) \rightarrow \mathbb{C}$ is a ϕ -dependent function, and introducing a running gauge-fixing parameter Λ_ρ , running masses $\mathbf{m}_\rho \equiv (m_\rho, m_{B\rho}, m_{u\rho}, m_{H\rho})$ and running coupling constants $\kappa_\rho, \lambda_{j\rho}$, we can achieve that $L_0 + z_\rho(L) - k_\rho$ has the same form as $L_0 + L$, in particular we absorb the novel bilinear interaction terms in the free Lagrangian:

$$(L_0^{\mathbf{m}, \Lambda} + z_\rho(L_{\kappa, \lambda}^{\mathbf{m}, \Lambda}) - k_\rho)(A, B, \varphi, u, \tilde{u}) = (L_0^{\mathbf{m}_\rho, \Lambda_\rho} + L_{\kappa_\rho, \lambda_\rho}^{\mathbf{m}_\rho, \Lambda_\rho})(A_\rho, B_\rho, \varphi_\rho, u, \tilde{u}). \quad (26)$$

We will use the shorthand notation

$$L_0 + z_\rho(L) - k_\rho = L_0^\rho + L^\rho$$

for this equation. Since every new field is of the form (25), the condition (26) is an equation for polynomials in the old fields; equating the coefficients the implicit definition (26) of the running quantities turns into the following explicit equations:

- for the wave functions

$$A_\rho^\mu = \sqrt{1 + a_{0\rho}} A^\mu, \quad B_\rho = \sqrt{1 + b_{0\rho}} B, \quad \varphi_\rho = \sqrt{1 + c_{0\rho}} \varphi; \quad (27)$$

- for the gauge-fixing parameter

$$\Lambda_\rho = \frac{\Lambda + a_{2\rho}}{1 + a_{0\rho}}; \quad (28)$$

- for the masses

$$\begin{aligned} m_\rho &= \sqrt{\frac{1 + a_{1\rho}}{1 + a_{0\rho}}} m, & m_{H\rho} &= \sqrt{\frac{1 + c_{1\rho}}{1 + c_{0\rho}}} m_H, \\ m_{B\rho} &= \sqrt{\frac{1 + b_{1\rho}}{1 + b_{0\rho}}} \frac{m}{\sqrt{\Lambda}}, & m_{u\rho} &= \sqrt{1 + c_{2\rho}} \frac{m}{\sqrt{\Lambda}}; \end{aligned} \quad (29)$$

- for the coupling constant

$$\kappa_\rho = \frac{1 + l_{0\rho}}{\sqrt{(1 + a_{0\rho})(1 + a_{1\rho})(1 + c_{0\rho})}} \kappa; \quad (30)$$

and the running coupling parameters λ_ρ are implicitly determined by

$$\begin{aligned} \frac{\kappa m}{\Lambda} \tilde{u}u\varphi &= \frac{\kappa_\rho \lambda_{10\rho} m_{u\rho}^2}{m_\rho} \tilde{u}u\varphi_\rho, \\ \kappa(1 + l_{1\rho}) B(A \cdot \partial\varphi) &= \kappa_\rho \lambda_{1\rho} B_\rho(A_\rho \cdot \partial\varphi_\rho), \\ &= \dots\dots\dots \\ \frac{\kappa(1 + l_{4\rho}) m_H^2}{m} B^2\varphi &= \frac{\kappa_\rho \lambda_{4\rho} m_{H\rho}^2}{m_\rho} B_\rho^2\varphi_\rho, \\ \kappa^2(1 + l_{5\rho}) (A \cdot A)\varphi^2 &= \kappa_\rho^2 \lambda_{5\rho} (A_\rho \cdot A_\rho)\varphi_\rho^2, \\ &= \dots\dots\dots \\ \kappa^2 l_{11\rho} (A \cdot A)^2 &= \kappa_\rho^2 \lambda_{11\rho} (A_\rho \cdot A_\rho)^2, \\ b_{2\rho} m (A \cdot \partial B) &= ((\lambda_{12\rho} - 1)m_\rho + \sqrt{\Lambda_\rho} m_{B\rho}) (A_\rho \cdot \partial B_\rho). \end{aligned} \quad (31)$$

The renormalizations (27)-(31) are not diagonal (as one naively might think): the new fields/parameters depend not only on the pertinent old field/parameter, because the coefficients $a_{j\rho}$, $b_{j\rho}$, $c_{j\rho}$, $l_{j\rho}$ are functions of the whole set $\{\mathbf{m}, \Lambda, \kappa\}$ of old parameters. The renormalization of the wave functions can be interpreted as follows: the field monomials appearing in $L_0 + z_\rho(L)$ can be viewed as a basis of a vector space. The redefinitions (27) are then a change of the “unit of length on the various coordinate axis”.

Remark 2.1 (“Perturbative agreement”). By the renormalization of the wave functions, masses and gauge fixing-parameter, we change the splitting of the total Lagrangian $L_0 + z_\rho(L)$ into a free and interacting part, i.e. we change the starting point for the perturbative expansion. To justify this, one has to show that the two perturbative QFTs given by the splittings $L_0 + z_\rho(L)$ and $L_0^\rho + L^\rho$, respectively, have the same physical content. This statement can be viewed as an application of the “Principle of Perturbative Agreement” of Hollands and Wald, which is used in [HW05] as an additional renormalization condition.

The proof that the “old” perturbative QFT (given by $L_0 + z_\rho(L)$) and the “new” one (given by $L_0^\rho + L^\rho$) are physically equivalent is beyond the scope of this paper. For the wave function and mass renormalization in a scalar field theory, the following conjecture has been formulated (by using the framework of algebraic QFT) and verified for a few examples [BDF]: given a renormalization prescription (i.e. an S -functional (16)) for the old perturbative QFT, there exists a renormalization prescription for the new perturbative QFT, such that the pertinent nets of local observables in the algebraic adiabatic limit (see [BF00] or [DF04, BDF09]) are equivalent. The corresponding isomorphisms can be chosen such that local fields are identified with local fields modulo the free field equation.

For models with spin 1 fields, the gauge-fixing parameter has also to be renormalized; and there is the difficulty that in general the new free theory (given by L_0^ρ) is *not* BRST-invariant, see Remark 7.2.

Geometrical interpretation at an arbitrary scale: since we have written the running Lagrangian $L_0 + z_\rho(L) - k_\rho$ in the form $L_0^\rho + L^\rho$, the equivalence of (7) and (6) can be applied to it: $L_0^\rho + L^\rho$ can be geometrically interpreted iff the $\lambda_{j\rho}$ have the values

$$\lambda_{1\rho} = \lambda_{2\rho} = \dots = \lambda_{9\rho} = 1, \quad \lambda_{11\rho} = \lambda_{12\rho} = 0. \quad (32)$$

To be precise: by ‘geometrical interpretation’ we mean here that

$$\begin{aligned} (L_0^{\mathbf{m}_\rho, \Lambda_\rho} + L_{\kappa_\rho, \lambda_\rho}^{\mathbf{m}_\rho, \Lambda_\rho})(A_\rho, B_\rho, \varphi_\rho, u, \tilde{u}) = & -\frac{1}{4} F_\rho^2 + \frac{1}{2} (D_\rho^\mu \Phi_\rho)^* D_{\rho\mu} \Phi_\rho - V_\rho(\Phi_\rho) \\ & + L_{\text{gf}}^\rho + L_{\text{ghost}}^\rho + \sqrt{\Lambda_\rho} m_{B\rho} \partial_\mu (A_\rho^\mu B_\rho), \end{aligned} \quad (33)$$

where $F_\rho^{\mu\nu} := \partial^\mu A_\rho^\nu - \partial^\nu A_\rho^\mu$,

$$\begin{aligned} \Phi_\rho & := iB_\rho + \frac{m_\rho}{\kappa_\rho} + \varphi_\rho, \quad D_\rho^\mu := \partial^\mu - i\kappa_\rho A_\rho^\mu \\ V_\rho(\Phi_\rho) & := \frac{\kappa_\rho^2 m_{H_\rho}^2}{8m_\rho^2} (\Phi_\rho^* \Phi_\rho)^2 - \frac{m_{H_\rho}^2}{4} (\Phi_\rho^* \Phi_\rho) + \frac{m_{H_\rho}^2 m_\rho^2}{8\kappa_\rho^2} \end{aligned} \quad (34)$$

and

$$\begin{aligned} L_{\text{gf}}^\rho & := -\frac{\Lambda_\rho}{2} \left(\partial \cdot A_\rho + \frac{m_{B\rho}}{\sqrt{\Lambda_\rho}} B_\rho \right)^2, \\ L_{\text{ghost}}^\rho & := \partial \tilde{u} \cdot \partial u - m_{u\rho}^2 \tilde{u} u - \frac{\kappa_\rho \lambda_{10\rho} m_{u\rho}^2}{m_\rho} \tilde{u} u \varphi_\rho. \end{aligned} \quad (35)$$

Our main question is whether (32) holds true when starting with the $U(1)$ -Higgs-model; for simplicity we also assume that initially we are in Feynman gauge: $\Lambda_{\rho=1} = 1$.

With these initial values, the geometrical interpretability (32) is equivalent to the following relations among the coefficients e_ρ :

$$\lambda_{1\rho} = 1 \text{ gives } \frac{1 + l_{1\rho}}{1 + l_{0\rho}} = \sqrt{\frac{1 + b_{0\rho}}{1 + a_{1\rho}}}, \quad (36)$$

$$\lambda_{2\rho} = 1 \text{ gives } l_{2\rho} = l_{1\rho}, \quad (37)$$

$$\lambda_{3\rho} = 1 \text{ gives } \frac{1 + l_{3\rho}}{1 + l_{0\rho}} = \frac{1 + c_{1\rho}}{1 + a_{1\rho}}, \quad (38)$$

$$\lambda_{4\rho} = 1 \text{ gives } \frac{1 + l_{4\rho}}{1 + l_{3\rho}} = \frac{1 + b_{0\rho}}{1 + c_{0\rho}}, \quad (39)$$

$$\lambda_{5\rho} = 1 \text{ gives } \frac{1 + l_{5\rho}}{(1 + l_{0\rho})^2} = \frac{1}{1 + a_{1\rho}}, \quad (40)$$

$$\lambda_{6\rho} = 1 \text{ gives } \frac{1 + l_{6\rho}}{1 + l_{5\rho}} = \frac{1 + b_{0\rho}}{1 + c_{0\rho}}, \quad (41)$$

$$\lambda_{7\rho} = 1 \text{ gives } \frac{1 + l_{7\rho}}{(1 + l_{0\rho})^2} = \frac{1 + c_{1\rho}}{(1 + a_{1\rho})^2}, \quad (42)$$

$$\lambda_{8\rho} = 1 \text{ gives } \frac{1 + l_{8\rho}}{1 + l_{7\rho}} = \frac{1 + b_{0\rho}}{1 + c_{0\rho}}, \quad (43)$$

$$\lambda_{9\rho} = 1 \text{ gives } \frac{1 + l_{9\rho}}{1 + l_{7\rho}} = \left(\frac{1 + b_{0\rho}}{1 + c_{0\rho}} \right)^2, \quad (44)$$

$$\lambda_{11\rho} = 0 \text{ gives } l_{11\rho} = 0, \quad (45)$$

$$\lambda_{12\rho} = 0 \text{ gives } b_{2\rho} = \sqrt{(1 + a_{2\rho})(1 + b_{1\rho})} - \sqrt{(1 + a_{1\rho})(1 + b_{0\rho})}. \quad (46)$$

Searching all values for the coefficients e_ρ which solve this system of equations, we find that this is quite a large set: neglecting k_ρ , 9 coefficients can freely be chosen (e.g. $a_{0\rho}$, $a_{1\rho}$, $a_{2\rho}$, $b_{0\rho}$, $b_{1\rho}$, $c_{0\rho}$, $c_{1\rho}$, $c_{2\rho}$ and $l_{0\rho}$), the other 11 coefficients are then uniquely determined by the 11 equations (36)-(46).

Combining the equations (38), (40) and (42) we obtain

$$\frac{1 + l_{7\rho}}{1 + l_{3\rho}} = \frac{1 + l_{5\rho}}{1 + l_{0\rho}}. \quad (47)$$

It will turn out that the conditions (46) and (47) are crucial for the geometrical interpretability.

For later purpose we mention that, with the considered initial values, the explicit formula for $\lambda_{10\rho}$ reads

$$\lambda_{10\rho} = \frac{1 + a_{1\rho}}{(1 + c_{2\rho})(1 + l_{0\rho})}. \quad (48)$$

3 BRST-invariance of the Lagrangian

The main result of this Section is that BRST-invariance of the Lagrangian is a sufficient (but not necessary) condition for the geometrical interpretation – for both, the initial Lagrangian $L_0 + L$ and the running Lagrangian $L_0 + z_\rho(L)$.

The BRST-transformation $s \equiv s_\beta = s_0 + \kappa\beta s_1$ is a graded derivation which commutes with partial derivatives and is given on the basic fields by

$$\begin{aligned} s A^\mu &= \partial^\mu u , & s B &= m u + \kappa\beta u \varphi , & s \varphi &= -\kappa\beta B u , \\ s u &= 0 , & s \tilde{u} &= -\Lambda \left(\partial A + \frac{m_B}{\sqrt{\Lambda}} B \right) . \end{aligned} \quad (49)$$

Since it is a priori not clear which of the coupling constant $\kappa, \kappa\lambda_j$ in L (2) is equal to the κ in the BRST-transformation, we have introduced the parameter β in s .

BRST-invariance of the initial Lagrangian: explicitly this property reads

$$s_\beta(L_0 + L) \simeq 0 \quad (50)$$

(where again \simeq means 'equal up to the addition of derivatives of local field polynomials'); it is equivalent to

$$\beta = 1 \quad \text{and the parameter values (6) and (11)}. \quad (51)$$

That the parameter values (51) imply (50) can be seen by formally interpreting the BRST-transformation of A^μ and (φ, B) as an infinitesimal gauge transformation,

$$s(A^\mu, \Phi) = \frac{d}{d\alpha} \Big|_{\alpha=0} (A^\mu + \alpha \partial^\mu u, e^{i\alpha\kappa u} \Phi) , \quad (52)$$

and by taking into account that D^μ is a pertinent covariant derivative. With that we immediately see that $s(F^2) = 0$, $s((D^\mu \Phi)^* D_\mu \Phi) = 0$, $s(V(\Phi)) = 0$, and by using (7) and a simple calculation we obtain

$$s(L_0 + L) = \sqrt{\Lambda} m_B s\partial(AB) + s(L_{\text{gf}} + L_{\text{ghost}}) = \partial_\mu (\sqrt{\Lambda} m_B s(A^\mu B) + (s\tilde{u})(sA^\mu)) . \quad (53)$$

The proof that (50) is also sufficient for the parameter values (51), is a straightforward calculation: inserting (1) and (2) into (50) one obtains (after some work) these parameter values. The relations (11) are precisely the condition that $s(L_{\text{gf}} + L_{\text{ghost}}) \simeq 0$, where we assume that $\beta = 1$ is already obtained from other parts of the calculation.

BRST-invariance of the running Lagrangian ($L_0 + z_\rho(L)$): the property

$$s_\beta(L_0 + z_\rho(L)) \simeq 0 , \quad (54)$$

where $s_\beta \equiv s$ is given by (49), determines $z_\rho(L) - k_\rho$ uniquely in terms of the three coefficients

$$a_\rho := a_{0\rho} , \quad b_\rho := b_{0\rho} , \quad l_\rho := c_{1\rho} , \quad (55)$$

which can freely be chosen. More explicitly, the condition (54) is equivalent to $\beta = 1$ and the following form of $z_\rho(L)$:

$$\begin{aligned}
z_\rho(L_{\kappa}^{\mathbf{m},\Lambda}) &\simeq k_\rho - \frac{1}{4}a_\rho F^2 + b_\rho \left(\frac{m^2}{2} A^2 - m A \partial B + \frac{1}{2} (\partial B)^2 + \frac{1}{2} (\partial \varphi)^2 \right) - l_\rho \frac{m_H^2}{2} \varphi^2 \\
&+ \kappa \left(-\frac{m}{\Lambda} \tilde{u} u \varphi + (1 + b_\rho) (m A^2 \varphi + B A \partial \varphi - \varphi A \partial B) \right. \\
&\quad \left. - \frac{(1 + l_\rho) m_H^2}{2m} (\varphi^3 + B^2 \varphi) \right) \\
&+ \kappa^2 \left(\frac{(1 + b_\rho)}{2} (A^2 \varphi^2 + A^2 B^2) - \frac{(1 + l_\rho) m_H^2}{8m^2} (\varphi^4 + 2 \varphi^2 B^2 + B^4) \right). \quad (56)
\end{aligned}$$

One verifies easily that with these relations among the coefficients e_ρ , the equations (36)-(46) are satisfied, that is, (54) implies indeed the geometrical interpretation (33). However, due to the presence of bilinear terms in $z_\rho(L)$, the difference between BRST-invariance of the Lagrangian (54) and the geometrical interpretation (33) does not only concern the ghost sector, as for $L_0 + L$ (see (11)), it is clearly bigger – the number of free coefficients e_ρ is 3 versus 9.

The proof that the set of solutions of the condition (54) is given by $\beta = 1$ and (56), is a somewhat lengthy and straightforward calculation, which is quite boring. More instructive is the following understanding of the parameter values (56): the above derivation (52)-(53) of BRST-invariance of $L_0 + L$, by using the geometrical interpretation, can only be applied to $L_0^\rho + L^\rho (= L_0 + z_\rho(L) - k_\rho)$, if the BRST transformation s (49) expressed in terms of the ρ -fields, has the same form as for the original fields, up to a global prefactor γ . Explicitly this requirement reads

$$\begin{aligned}
s A_\rho^\mu &= \sqrt{1 + a_{0\rho}} s A^\mu = \gamma \partial^\mu u, \\
s B_\rho &= \sqrt{1 + b_{0\rho}} s B = \gamma (m_\rho u + \kappa_\rho u \varphi_\rho), \\
s \varphi_\rho &= \sqrt{1 + c_{0\rho}} s \varphi = -\gamma (\kappa_\rho B_\rho u) \\
s \tilde{u} &= -\gamma \Lambda_\rho (\partial A_\rho + \frac{m_{B\rho}}{\sqrt{\Lambda_\rho}} B_\rho) \quad (57)
\end{aligned}$$

and $s u = \gamma 0 = 0$ is trivially satisfied. From the first equation we obtain

$$\gamma = \sqrt{1 + a_{0\rho}} \quad (58)$$

and with that the further equations are equivalent to

$$b_\rho := b_{0\rho} = a_{1\rho} = c_{0\rho} = l_{0\rho} \quad \text{and} \quad b_{1\rho} = 0 = a_{2\rho}. \quad (59)$$

To take the demand for validity of the geometrical interpretation into account, we insert (59) into (36)-(46), this yields

$$\begin{aligned}
b_\rho &= l_{1\rho} = l_{2\rho} = l_{5\rho} = l_{6\rho} = -b_{2\rho} \\
l_\rho &:= c_{1\rho} = l_{3\rho} = l_{4\rho} = l_{7\rho} = l_{8\rho} = l_{9\rho} \\
l_{11\rho} &= 0. \quad (60)
\end{aligned}$$

In addition, the derivation (52)-(53) needs BRST-invariance of $(L_{\text{gf}}^\rho + L_{\text{ghost}}^\rho)$ (35), which is equivalent to $m_{u\rho}^2 = \frac{m_{B\rho} m_\rho}{\sqrt{\Lambda_\rho}}$ and $\lambda_{10\rho} = 1$ (similarly to (11)); both equations give

$$c_{2\rho} = 0 \tag{61}$$

by using (59) (and the formulas (28), (29), (48) for the running quantities). The parameter relations (59), (60) and (61) agree precisely with (56).

As usual, the BRST-transformation s (49) is nilpotent modulo the field equations of $L_0 + L$; we point out that this holds also for s expressed in terms of the ρ -fields (i.e. (57)-(58)) w.r.t. the field equations of the new Lagrangian $L_0^\rho + L^\rho$:

$$s^2 \tilde{u} = -\Lambda_\rho(1 + a_\rho) \left(\square u + \frac{m_{B\rho}}{\sqrt{\Lambda_\rho}} (m_\rho u + \kappa_\rho u \varphi_\rho) \right) = \Lambda_\rho(1 + a_\rho) \frac{\delta \int dx (L_0^\rho + L^\rho)}{\delta \tilde{u}}, \tag{62}$$

where we use the preceding relations, i.e. we assume that (54) holds true.

4 Perturbative gauge invariance (PGI)

For the initial model $S(iL(g))$ we admit all renormalization prescriptions which fulfill the Epstein-Glaser axioms [EG73, DF04] and a suitable version of BRST-invariance. The latter should be well adapted to the inductive Epstein-Glaser construction of the time-ordered products and to our definition of the RG-flow. We will see that PGI [DHKS94, DS99] fulfills these criteria.

Physical consistency (PC). To motivate PGI we start with PC, which is a somewhat weaker condition [KO79, DS00, Gri00]. Let Q be the charge implementing the free BRST-transformation $s_0 := s|_{\kappa=0}$, explicitly

$$[Q, \phi]_\star^\mp \approx i\hbar s_0 \phi, \quad \phi = A^\mu, B, \varphi, u, \tilde{u}, \tag{63}$$

where $[\cdot, \cdot]_\star^\mp$ denotes the graded commutator w.r.t. the \star -product and \approx means 'equal modulo the free field equations'. The nilpotency $Q^2 \approx 0$ reflects $s_0^2 \approx 0$. For our model with Feynman gauge $\Lambda = 1$, the charge Q is given by the somewhat heuristic formula⁶

$$Q = \int_{x^0=\text{constant}} d^3x \left((\partial A + mB) \partial^0 u - \partial^0 (\partial A + mB) u \right). \tag{64}$$

For the asymptotic free fields, the "subspace" of physical states can be described as

$$\mathcal{H}_{\text{phys}} := \frac{\ker Q}{\text{ran } Q}.$$

The operator $\mathbf{S}[L]$ (17) induces a well defined operator from $\mathcal{H}_{\text{phys}}$ into itself iff

$$[Q, \mathbf{S}[L]]_\star|_{\ker Q} \approx 0,$$

⁶A rigorous definition of Q is given in [DF99].

see e.g. [DS00]. This is the reason to require

$$\text{physical consistency (PC): } 0 \approx [Q, \mathbf{S}[L]]_\star \equiv \lim_{\varepsilon \downarrow 0} [Q, S(iL(g_\varepsilon)/\hbar)]_\star . \quad (65)$$

Perturbative gauge invariance (PGI): to fulfill PC in the inductive Epstein-Glaser construction of time-ordered products, we need a version of PC *before* the adiabatic limit $g \rightarrow 1$ is taken: precisely for this purpose PGI was introduced in [DHKS94].

PGI is the condition that to a given local interaction

$$\mathcal{L}(g) := \hbar^{-1} \sum_{k=1}^{\infty} \kappa^k \int dx (g(x))^k \mathcal{L}_{(k)}(x) , \quad (66)$$

there exists a “ Q -vertex”

$$\mathcal{P}^\nu(g; f) := \sum_{k=1}^{\infty} \kappa^k \int dx (g(x))^{(k-1)} \mathcal{P}_{(k)}^\nu(x) f(x) \quad (67)$$

(where $g, f \in \mathcal{D}(\mathbb{R}^4)$ and $\mathcal{L}_{(k)}, \mathcal{P}_{(k)}$ are local field polynomials) and a renormalization of the time-ordered products such that

$$[Q, S(i\mathcal{L}(g))]_\star \approx \frac{d}{d\eta} \Big|_{\eta=0} S(i\mathcal{L}(g) + \eta \mathcal{P}^\nu(g; \partial_\nu g)) . \quad (68)$$

That PGI implies PC, is easy to see (on the heuristic level on which we treat the adiabatic limit in this paper): the r.h.s. of (68) vanishes in the adiabatic limit, since it is linear in the Q -vertex, the latter is linear in $\partial_\nu g$ and $\partial_\nu g_\varepsilon \sim \varepsilon$.

For time-ordered products T_n of order $n \geq 2$, PGI is a renormalization condition – it is a particular case of the ‘Master BRST Identity’, which is the application of the ‘Master Ward Identity’ to the conservation of the free BRST-current, see [DB02, DF03].

It is well-known that the $U(1)$ -Higgs model is anomaly-free. Hence, our initial model can be renormalized such that PGI (68) holds to all orders in κ , where $\mathcal{L}(g) := L(g)$ is given by (13) and

$$\mathcal{P}^\nu(g; f) := \int dx \left(\kappa P_1^\nu(x) + \kappa^2 g(x) P_2^\nu(x) \right) f(x), \quad (69)$$

with

$$P_1^\nu = m A^\nu u \varphi - \partial^\nu B u \varphi + B u \partial^\nu \varphi , \quad P_2^\nu = A^\nu u \varphi^2 + A^\nu u B^2 , \quad (70)$$

where L_k (13) and P_k^ν are $(k+2)$ -linear in the basic fields and Feynman gauge $\Lambda = 1$ is chosen.

To apply PGI to the running interaction $z_\rho(L)$, we insert the power series (20) for the coefficients e_ρ into $z_\rho(L)$ (19), to write the latter as a power series in κ ,

$$z_\rho(L) = \hbar^{-1} \sum_{k=1}^{\infty} z_{\rho k}(L) \kappa^k . \quad (71)$$

So, for this interaction, $\mathcal{L}(g)$ (66) is given by

$$\mathcal{L}(g) = z_\rho(L)(g) := \hbar^{-1} \sum_{k=1}^{\infty} \int dx z_{\rho k}(L)(x) (\kappa g(x))^k . \quad (72)$$

Explicitly, with $\Lambda_{\rho=1} = 1$ we have

$$\mathcal{L}_{(1)} = L_1 , \quad \mathcal{L}_{(2)} = L_2 + \hbar L_0^{(1)} , \quad \mathcal{L}_{(3)} = \hbar L_1^{(1)} , \quad \mathcal{L}_{(4)} = \hbar L_2^{(1)} + \hbar^2 L_0^{(2)} \quad (73)$$

etc., where $L_k^{(j)}$ is $(k+2)$ -linear in the basic fields and the upper index j denotes the order in \hbar ; explicitly

$$\begin{aligned} L_0^{(j)} &= -\frac{1}{4} a_{0\rho}^{(j)} F^2 + \frac{m^2}{2} a_{1\rho}^{(j)} A^2 - \frac{a_{2\rho}^{(j)}}{2} (\partial A)^2 + \frac{1}{2} b_{0\rho}^{(j)} (\partial B)^2 - \frac{m^2}{2} b_{1\rho}^{(j)} B^2 \\ &\quad + \frac{1}{2} c_{0\rho}^{(j)} (\partial\varphi)^2 - \frac{m_H^2}{2} c_{1\rho}^{(j)} \varphi^2 - m^2 c_{2\rho}^{(j)} \tilde{u}u + m b_{2\rho}^{(j)} A\partial B , \\ L_1^{(j)} &= m l_{0\rho}^{(j)} A^2\varphi + l_{1\rho}^{(j)} B(A\partial\varphi) - l_{2\rho}^{(j)} \varphi(A\partial B) - \frac{m_H^2}{2m} (l_{3\rho}^{(j)} \varphi^3 + l_{4\rho}^{(j)} B^2\varphi) , \\ L_2^{(j)} &= \frac{1}{2} \left(l_{5\rho}^{(j)} A^2\varphi^2 + l_{6\rho}^{(j)} A^2B^2 \right) - \frac{m_H^2}{8m^2} (l_{7\rho}^{(j)} \varphi^4 + 2l_{8\rho}^{(j)} \varphi^2B^2 + l_{9\rho}^{(j)} B^4) + l_{11\rho}^{(j)} (A^2)^2 \end{aligned} \quad (74)$$

for $j \geq 1$. The pertinent $\mathcal{P}_{(k)}$ in (67) must have a similar structure

$$\mathcal{P}_{(1)} = P_1 , \quad \mathcal{P}_{(2)} = P_2 + \hbar P_0^{(1)} , \quad \mathcal{P}_{(3)} = \hbar P_1^{(1)} , \quad \mathcal{P}_{(4)} = \hbar P_2^{(1)} + \hbar^2 P_0^{(2)} \quad (75)$$

etc., where the indices of $P_k^{(j)}$ have the same meaning as for $L_k^{(j)}$.

5 Stability of physical consistency under the renormalization group flow

Stability of PC: it is hard to find out whether PGI is maintained under the RG-flow, i.e. whether PGI for $\mathcal{L}(g) = L(g)$ (13) implies PGI for $\mathcal{L}(g) = z_\rho(L)(g)$ (72). In Sect. 7 we show that PGI for $S(iz_\rho(L)(g))$ can be fulfilled on the level of tree diagrams (with vertices $z_\rho(L)(g)$), if one takes only the 1-loop contributions $e_\rho^{(1)}$ (20) to $z_\rho(L)$ into account. But this depends on the renormalization prescription for $S(iL(g))$: using a prescription corresponding to the minimal subtraction scheme, PGI gets lost under the RG-flow, already at the level of tree diagrams.

However, the somewhat weaker property of PC is maintained under the RG-flow; more precisely we will prove that

$$[Q, \mathbf{S}[L]]_\star \approx 0 \quad \Rightarrow \quad [Q, \mathbf{S}[z_\rho(L)]]_\star \approx 0 . \quad (76)$$

Hence, at least in this weak form, BRST-invariance of the time-ordered products is stable under the RG-flow. We point out that (76) is a model-independent result; only rather weak assumptions are needed, which will be given in the course of the proof.

Construction of $z_\rho(L)$: to prove (76), we need to understand precisely how $z_\rho(L)$ is constructed. We use the formalism of [DF04] (see also [BDF09]), in particular we apply the Main Theorem [DF04, HW03]: assuming that S fulfills the axioms of Epstein-Glaser renormalization, this holds also for the scaled time-ordered products $\sigma_\rho \circ S \circ \sigma_\rho^{-1}$; therefore, there exists a unique map $Z_\rho \equiv Z_{\rho, \mathbf{m}}$ from the space of local interactions into itself such that ⁷

$$\sigma_\rho \circ S_{\rho^{-1} \mathbf{m}} \circ \sigma_\rho^{-1} = S_{\mathbf{m}} \circ Z_{\rho, \mathbf{m}} \quad (77)$$

(the lower index \mathbf{m} on S and Z_ρ denotes the masses of the underlying \star -product, i.e. the masses of the Feynman propagators).

In addition Z_ρ is of the following form [DF04, Prop. 4.3]: let \mathcal{P} be the space of local field polynomials, $h \in \mathcal{D}(\mathbb{R}^4)$ and $A(h) = \int dx A(x) h(x)$ for $A \in \mathcal{P}$. Given Z_ρ , there exist linear and symmetric maps $d_{n,a}^\rho : \mathcal{P}^{\otimes n} \rightarrow \mathcal{P}$ for $n \geq 2$, $a \equiv (a_1, \dots, a_n) \in (\mathbb{N}_0^4)^n$, such that

$$Z_\rho(A(h)) = A(h) + \sum_{n=2}^{\infty} \frac{1}{n!} \sum_a \int dx d_{n,a}^\rho(A^{\otimes n})(x) \prod_{l=1}^n (\partial^{a_l} h(x)) . \quad (78)$$

The expressions $d_{n,a}^\rho(A^{\otimes n})$ are uniquely determined if one requires $d_{n,a}^\rho(A^{\otimes n}) \in \mathcal{P}_{\text{bal}}$, where $\mathcal{P}_{\text{bal}} \subset \mathcal{P}$ is the subspace of “balanced fields”, defined in [DF04].

Applying (78) to $A(h) = iL(g)/\hbar = (i/\hbar) \sum_{j=1,2} \int dx (\kappa g(x))^j L_j(x)$ (13), we get

$$\begin{aligned} Z_\rho(iL(g)/\hbar) &= iL(g)/\hbar + \sum_{n=2}^{\infty} \frac{i^n}{n! \hbar^n} \sum_a \sum_{j_1, \dots, j_n=1,2} \kappa^{j_1 + \dots + j_n} \\ &\quad \cdot \int dx d_{n,a}^\rho(L_{j_1} \otimes \dots \otimes L_{j_n})(x) \prod_{l=1}^n \partial^{a_l} (g(x))^{j_l} . \end{aligned} \quad (79)$$

In view of the adiabatic limit and $\partial g_\varepsilon(x) = \mathcal{O}(\varepsilon)$, we cut off the terms with derivatives of g :

$$Z_\rho(iL(g_\varepsilon)/\hbar) = i z_\rho(L)(g_\varepsilon) + \mathcal{O}(\varepsilon) , \quad (80)$$

where

$$z_\rho(L)(g) = \frac{1}{\hbar} \left(L(g) + \sum_{n=2}^{\infty} \frac{i^{n-1}}{n! \hbar^{n-1}} \sum_{j_i=1,2} \int dx d_{n,0}^\rho(L_{j_1} \otimes \dots \otimes L_{j_n})(x) (\kappa g(x))^{j_1 + \dots + j_n} \right) . \quad (81)$$

Hence, $z_\rho(L)(g)$ is indeed of the form (72) with

$$z_{\rho k}(L) = L_k + \sum_{n=2}^k \frac{i^{n-1}}{n! \hbar^{n-1}} \sum_{j_1 + \dots + j_n = k} d_{n,0}^\rho(L_{j_1} \otimes \dots \otimes L_{j_n}) , \quad (82)$$

⁷We use the convention for Z_ρ given in [BDF09], which differs by factors i from the definition $\tilde{Z}_\rho(F) := D_\rho(e_\otimes^F)$ in [DF04], namely: $Z_\rho(iF) = i \tilde{Z}_\rho(F)$.

where $L_k := 0$ for $k \geq 3$. Finally, $z_\rho(L)$ is obtained from (82) by means of (71). From (72) and (80) we see that $z_{\rho k}(L)$ is uniquely determined up to the addition of terms $\partial^a A$, $|a| \geq 1$, $A \in \mathcal{P}$ – as claimed in (19).

From (80) and (multi-)linearity of the time-ordered products, we conclude that the adiabatic limit of (77) applied to $iL(g)$ gives indeed (18):

$$\begin{aligned} \sigma_\rho(\mathbf{S}_{\rho^{-1}\mathbf{m}}[\sigma_\rho^{-1}(L^{\mathbf{m}})]) &:= \lim_{\varepsilon \downarrow 0} \sigma_\rho \circ S_{\rho^{-1}\mathbf{m}} \circ \sigma_\rho^{-1}(iL^{\mathbf{m}}(g_\varepsilon)) = \lim_{\varepsilon \downarrow 0} S_{\mathbf{m}}(Z_\rho(iL^{\mathbf{m}}(g_\varepsilon))) \\ &= \lim_{\varepsilon \downarrow 0} S_{\mathbf{m}}(i z_\rho(L^{\mathbf{m}})(g_\varepsilon)) =: \mathbf{S}_{\mathbf{m}}[z_\rho(L^{\mathbf{m}})] . \end{aligned} \quad (83)$$

Here we assume that $S_{\mathbf{m}}(iL(g))$ is renormalized such that the adiabatic limit $\varepsilon \downarrow 0$ exists and is unique for $\sigma_\rho \circ S_{\rho^{-1}\mathbf{m}} \circ \sigma_\rho^{-1}(iL(g_\varepsilon))$ for all $\rho > 0$; hence, this limit exists also for $S(i z_\rho(L)(g_\varepsilon))$.

Proof of stability of PC (76): by using the Main Theorem, the relations

$$\sigma_\rho^{-1}(L^{\mathbf{m}}(g)) = L^{\rho^{-1}\mathbf{m}}(g_{1/\rho}) \quad (\text{again } g_\lambda(x) := g(\lambda x)) \quad (84)$$

and

$$\sigma_\rho(F \star_{\rho^{-1}\mathbf{m}} G) = \sigma_\rho(F) \star_{\mathbf{m}} \sigma_\rho(G) , \quad \rho \sigma_\rho \circ Q_{\rho^{-1}\mathbf{m}} = Q_{\mathbf{m}} , \quad (85)$$

we obtain

$$\begin{aligned} [Q_{\mathbf{m}}, S_{\mathbf{m}}(Z_\rho(iL^{\mathbf{m}}(g_\varepsilon)))]_{\star_{\mathbf{m}}} &= [Q_{\mathbf{m}}, \sigma_\rho \circ S_{\rho^{-1}\mathbf{m}}(iL^{\rho^{-1}\mathbf{m}}(g_{\varepsilon/\rho}))]_{\star_{\mathbf{m}}} \\ &= \rho \sigma_\rho \left([Q_{\rho^{-1}\mathbf{m}}, S_{\rho^{-1}\mathbf{m}}(iL^{\rho^{-1}\mathbf{m}}(g_{\varepsilon/\rho}))]_{\star_{\rho^{-1}\mathbf{m}}} \right) . \end{aligned} \quad (86)$$

Now, assuming that $S_{\mathbf{m}}(iL^{\mathbf{m}}(g))$ fulfills PC (65) for all values $m, m_H > 0$ of the masses, we conclude that the adiabatic limit $\varepsilon \downarrow 0$ of the last expression in (86) vanishes. (Here we use that it does not matter whether we perform the adiabatic limit with g or $g_{1/\rho}$, since it is unique.) With that and with (80) we obtain the assertion (76):

$$0 \approx \lim_{\varepsilon \downarrow 0} [Q, S(Z_\rho(iL(g_\varepsilon)))]_{\star} = \lim_{\varepsilon \downarrow 0} [Q, S(i z_\rho(L)(g_\varepsilon))]_{\star} = [Q, \mathbf{S}[z_\rho(L)]]_{\star} . \quad \square \quad (87)$$

Completion of the derivation of the form of $z_\rho(L)$: having given the construction of $z_\rho(L)$ (82) (see also Sect. 5 of [DF04]), we are able to explain why on the r.h.s. of (19) precisely these field monomials appear and no others:

- each term appearing in $z_\rho(L)$ is Lorentz invariant, has ghost number = 0 and its mass dimension is ≤ 4 (see formula (5.5) in [DF04]).
- Since the only interaction term containing $\tilde{u}u$ is $m\tilde{u}u\varphi$, each term in $(z_\rho(L) - L)$ which is bilinear in the ghost fields has a factor m^2 and, hence, its mass dimension is ≤ 2 . This excludes a $\partial\tilde{u}\partial u$ -term and non-trivial trilinear and quadrilinear terms containing $\tilde{u}u$.
- The property that L is even under the field parity transformation (3) goes over to each diagram contributing to $S(iL(g))$ and, hence, each term appearing in $z_\rho(L)$ has also this property. This reduces the number of possible terms in $z_\rho(L)$ quite strongly.

- To discuss the appearance of one-leg terms $\sum_a c_a \partial^a \phi$ in $z_\rho(L)$ (where $\phi = A^\mu, B, \varphi$ and $c_a \in \mathbb{C}$), we write (77) to n -th order, by using the chain rule:

$$\begin{aligned} Z_{\rho, \mathbf{m}}^{(n)}(L(g_\varepsilon)^{\otimes n}) &= \sigma_\rho \circ T_{n \mathbf{m}/\rho}((\sigma_\rho^{-1} L(g_\varepsilon))^{\otimes n}) - T_{n \mathbf{m}}(L(g_\varepsilon)^{\otimes n}) \\ &\quad - \sum_{P \in \text{Part}(\{1, \dots, n\}, n > |P| > 1)} T_{|P| \mathbf{m}}(\otimes_{I \in P} Z_{\rho, \mathbf{m}}^{|I|}(L(g_\varepsilon)^{\otimes |I|})) , \end{aligned} \quad (88)$$

where $Z_\rho^{(n)} := Z_\rho^{(n)}(0)$ is the n -th derivative of $Z_\rho(F)$ at $F = 0$ and the two terms with $|P| = n$ and $|P| = 1$, resp., are explicitly written out. Taking (79) into account, we see that each one-leg term appearing on the r.h.s. of (88) is a sum of terms of the form

$$\int dx_1 \dots dx_k G_1(\varepsilon x_1) \dots G_k(\varepsilon x_k) \sum_b \partial^b \phi(x_k) t_b(x_1 - x_k, \dots, x_{k-1} - x_k) , \quad (89)$$

where $k = n$ or $k = |P|$, the testfunctions G_j are of the form $G_j(x) = \prod_{l=1}^{n_j} \partial^{a_{jl}} g(x)$ and $t_a = \omega_0(T_k(\dots))$ is the vacuum expectation value of a time-ordered product. The expression (89) can be written as an integral in momentum space: up to a power of (2π) as prefactor it is equal to

$$\begin{aligned} &\int dp_1 \dots dp_k \hat{G}_1(p_1) \dots \hat{G}_k(p_k) \hat{\phi}(-\varepsilon(p_1 + \dots + p_k)) \\ &\quad \cdot \sum_b (-i\varepsilon(p_1 + \dots + p_k))^b \hat{t}_b(-\varepsilon p_1, \dots, -\varepsilon p_{k-1}) . \end{aligned} \quad (90)$$

From [EG73] we know that $\hat{t}_b(p)$ is analytic in a neighbourhood of $p = 0$, since all fields are massive. Hence, in the adiabatic limit $\varepsilon \downarrow 0$ of (90), the $(|b| > 0)$ -terms vanish and, hence, do not contribute to $z_\rho(L)$.

To avoid the appearance of a $(b = 0)$ -term in $z_\rho(L)$, we first mention that we only have to consider the case in which the singular order of $t := t_{b=0}$ is $\omega(t) \geq 0$,⁸ for the following reason: a term with $\omega(t) < 0$ is non-local, i.e. $\text{supp}(t) \not\subset \{0\}$. But the l.h.s. of (88) is local; hence, the $(\omega(t) < 0)$ -terms appearing on the r.h.s. of (88) must cancel, when restricted to $\mathcal{D}(\mathbb{R}^{4k} \setminus \Delta_k)$ (where $\Delta_k := \{(x_1, \dots, x_k) \in \mathbb{R}^{4k} \mid x_1 = \dots = x_k\}$). Since for these terms, the extension to $\mathcal{D}(\mathbb{R}^{4k})$ is unique, they cancel also on $\mathcal{D}(\mathbb{R}^{4k})$.

Obviously, the finite renormalization

$$\hat{t}(p) \mapsto \hat{t}(p) - \hat{t}(0) , \quad t \equiv t_{b=0} , \quad (91)$$

⁸ For $t \in \mathcal{D}'(\mathbb{R}^l)$ or $t \in \mathcal{D}'(\mathbb{R}^l \setminus \{0\})$, the singular order is defined as $\omega(t) := \text{sd}(t) - l$, where $\text{sd}(t)$ is Steinmann's scaling degree of t , which measures the UV-behaviour of t [Ste71]. In the Epstein-Glaser framework, renormalization is the extension of a distribution $t^\circ \in \mathcal{D}'(\mathbb{R}^l \setminus \{0\})$ to a distribution $t \in \mathcal{D}'(\mathbb{R}^l)$, with the condition that $\text{sd}(t) = \text{sd}(t^\circ)$. In the case $\text{sd}(t^\circ) < l$, the extension is unique, due to the scaling degree requirement, and obtained by "direct extension", see [BF00, Theorem 5.2], [DF04, Appendix B] and [DFKR14, Theorem 4.1].

which is admitted due to $\omega(t) \geq 0$, removes the possible one-leg terms in $z_\rho(L)$. This renormalization preserves PGI, because of

$$[Q, \phi] \approx i \partial_\mu \phi_1^\mu, \text{ where } \phi_1^\mu = 0, -\frac{\partial^\mu u}{m}, g^{\mu\nu} u \text{ for } \phi = \varphi, B, A^\nu, \text{ resp.;} \quad (92)$$

in detail:

$$[Q, \hat{t}(0) \phi(x_k) \delta(x_1 - x_k, \dots, x_{k-1} - x_k)] \approx i \sum_{l=1}^k \partial_\mu^{x_l} \left(\hat{t}(0) \phi_1^\mu(x_k) \delta(x_1 - x_k, \dots, x_{k-1} - x_k) \right). \quad (93)$$

We point out that when we perform the finite renormalization (91) for a t belonging to $T_{n\mathbf{m}}$, then the corresponding t belonging to $\sigma_\rho \circ T_{n\mathbf{m}/\rho} \circ (\sigma_\rho^{-1})^{\otimes n}$ is automatically modified by precisely the same finite renormalization, because the renormalization condition $\hat{t}(0) = 0$ is scaling invariant.

If one does not perform the finite renormalization (91), one-leg terms may appear in $z_\rho(L)$; however, only in second and higher loop orders. Namely, they fulfill (23) with $q_0 = 0$ and $n := r_0 + s \geq 1$, hence they appear in (19) as

$$z_\rho(L) = \hbar^{-1} \left(\kappa^{-1} \sum_{n=2}^{\infty} e_\rho^{(n)} (\kappa^2 \hbar)^n \phi + \dots \right). \quad (94)$$

6 Geometrical interpretation at all scales to 1-loop order

In this section we explain, how one can fulfill the geometrical interpretation at all scales, i.e. the equations (36)-(46), on 1-loop level. For this purpose we derive a lot of results about the 1-loop coefficients $e_\rho^{(1)}$ (20) of the running interaction $z_\rho(L)$ (19). Throughout we choose Feynman gauge $\Lambda = 1$ for the initial $U(1)$ -Higgs model. The conventions for the signs and factors $i, 2\pi$ are fixed in (207).

6.1 The two ways to renormalize

Renormalizing a 1-loop Feynman diagram, there are two crucially different methods to choose the renormalization mass scale. We explain this in terms of the computation of the 1-loop coefficient $c_{2\rho}^{(1)}$, which is the one that is most easily to compute.

Computation of $c_{2\rho}^{(1)}$: we recall that $Z_\rho(iL(g)/\hbar)$ is a formal Taylor series,

$$Z_\rho(iL(g)/\hbar) = iL(g)/\hbar + \sum_{n=2}^{\infty} \frac{i^n}{\hbar^n n!} Z_\rho^{(n)}(L(g)^{\otimes n}); \quad (95)$$

according to (88) the ($n = 2$)-term is obtained by

$$Z_\rho^{(2)}(L^{\mathbf{m}}(g)^{\otimes 2}) = \sigma_\rho \circ T_{2\rho^{-1}\mathbf{m}}(\sigma_\rho^{-1}(L^{\mathbf{m}}(g))^{\otimes 2}) - T_{2\mathbf{m}}(L^{\mathbf{m}}(g)^{\otimes 2}). \quad (96)$$

To compute $c_{2\rho}^{(1)}$ we select the term with external legs $\tilde{u}u$, which is $\sim \kappa^2$:

$$T_{2\mathbf{m}}(L_1(x_1) \otimes L_1(x_2)) = m^2 (t_{\mathbf{m}}^{\tilde{u}u}(x_1 - x_2) \tilde{u}(x_1)u(x_2) + (x_1 \leftrightarrow x_2) + \dots) , \quad (97)$$

where

$$t_{\mathbf{m}}^{\tilde{u}u}(x_1 - x_2) := \omega_0(T_{2\mathbf{m}}(u(x_1)\varphi(x_1) \otimes \tilde{u}(x_2)\varphi(x_2))) \quad (98)$$

and ω_0 denotes the vacuum state. The corresponding contribution to $Z_\rho(\hbar^{-1}iL(g))$ reads

$$\begin{aligned} Z_\rho(iL(g)/\hbar) &= \hbar^{-1}iL(g) - \frac{\kappa^2 m^2}{\hbar^2} \int dx_1 dx_2 g(x_1)g(x_2) \\ &\cdot (\rho^4 t_{\rho^{-1}\mathbf{m}}^{\tilde{u}u}(\rho(x_1 - x_2)) - t_{\mathbf{m}}^{\tilde{u}u}(x_1 - x_2)) \tilde{u}(x_1)u(x_2) + \dots . \end{aligned} \quad (99)$$

We will see that

$$\rho^4 t_{\rho^{-1}\mathbf{m}}^{\tilde{u}u}(\rho y) - t_{\mathbf{m}}^{\tilde{u}u}(y) = \hbar^2 C_{\text{fish}} \log \rho \delta(y) \quad (100)$$

with a constant $C_{\text{fish}} \in i\mathbb{R}$. Inserting (100) into (99) and using (79), (82), we end up with

$$c_{2\rho}^{(1)} = -iC_{\text{fish}} \log \rho . \quad (101)$$

To derive (100) and to compute the number C_{fish} , we start with the unrenormalized version of $t_{\mathbf{m}}^{\tilde{u}u}$: the restriction of $t_{\mathbf{m}}^{\tilde{u}u}(y)$ to $\mathcal{D}(\mathbb{R}^4 \setminus \{0\})$ agrees with

$$t_{\mathbf{m}}^{\tilde{u}u \circ}(y) := \hbar^2 t_{m, m_H}(y) \quad t_{m, m_H}(y) := \Delta_m^F(y) \Delta_{m_H}^F(y) \in \mathcal{D}'(\mathbb{R}^4 \setminus \{0\}) , \quad (102)$$

where Δ_m^F is the Feynman propagator to the mass m . Due to $\rho^2 \Delta_{\rho^{-1}m}^F(\rho y) = \Delta_m^F(y)$, the unrenormalized distribution $t_{\mathbf{m}}^{\tilde{u}u \circ}$ scales homogeneously,

$$\rho^4 t_{\rho^{-1}\mathbf{m}}^{\tilde{u}u \circ}(\rho y) = t_{\mathbf{m}}^{\tilde{u}u \circ}(y) . \quad (103)$$

The question is, whether this property can be maintained in the process of renormalization (i.e. extension, see footnote 8).

To construct the extension $t_{\mathbf{m}}^{\tilde{u}u} \in \mathcal{D}'(\mathbb{R}^4)$ we use the scaling and mass expansion (shortly 'sm-expansion') [Düt15]; in the present case this means that we split $t_{\mathbf{m}}^{\tilde{u}u \circ}(y)$ into the corresponding massless distribution $-\hbar^2 t_{\text{fish}}^\circ(y)$ and a remainder $r_{\mathbf{m}}^\circ(y)$, which is of order $r_{\mathbf{m}}^\circ = \mathcal{O}(m^2, m_H^2)$:

$$t_{\mathbf{m}}^{\tilde{u}u \circ}(y) = \hbar^2 t_{\text{fish}}^\circ(y) + r_{\mathbf{m}}^\circ(y) , \quad t_{\text{fish}}^\circ(y) := (D^F(y))^2 , \quad \text{sd}(t_{\text{fish}}^\circ) = \text{sd}(r_{\mathbf{m}}^\circ) = 4 , \quad (104)$$

where $D^F := \Delta_{m=0}^F$ is the massless Feynman propagator. The remainder $r_{\mathbf{m}}^\circ$ has a unique extension $r_{\mathbf{m}} \in \mathcal{D}'(\mathbb{R}^4)$ with $\text{sd}(r_{\mathbf{m}}) = \text{sd}(r_{\mathbf{m}}^\circ) = 2$, which is obtained by direct extension; it preserves the homogeneous scaling (103).

The unrenormalized massless part t_{fish}° scales homogeneously in y , but this property cannot be preserved: the extension needs a mass scale $M > 0$ and with that homogeneous scaling in y is broken at least by a logarithmic term. All extensions with such a minimal breaking can be obtained by differential renormalization:

$$t_{\text{fish}}^M(y) = \frac{-1}{64\pi^4} \square_y \left(\frac{\log(-M^2(y^2 - i0))}{y^2 - i0} \right) \in \mathcal{D}'(\mathbb{R}^4) , \quad M > 0 \text{ arbitrary} , \quad (105)$$

see e.g. appendix B in [DF04].

The two methods to choose the renormalization mass scale. Whether homogeneous scaling in y and (m^{-1}, m_H^{-1}) (103) is maintained depends on the following choice:

- (A) If we choose for M a fixed mass scale, which is independent of m, m_H , homogeneous scaling is broken:

$$\begin{aligned} \rho^4 t_{\rho^{-1}\mathbf{m}}^{\tilde{u}u}(\rho y) - t_{\mathbf{m}}^{\tilde{u}u}(y) &= \hbar^2 (\rho^4 t_{\text{fish}}^M(\rho y) - t_{\text{fish}}^M(y)) \\ &= \hbar^2 C_{\text{fish}} \log \rho \delta(y), \quad C_{\text{fish}} := \frac{-i}{8\pi^2}, \end{aligned} \quad (106)$$

by using $\square(\frac{1}{y^2-i0}) = i4\pi^2 \delta(y)$. The breaking term is unique, i.e. independent of M ; therefore, we may admit different values of M for different t -distributions, however, all M 's must be independent of m, m_H .

- (B) Homogeneous scaling (103) can be maintained by choosing $M := \alpha_1 m + \alpha_2 m_H$, where $(\alpha_1, \alpha_2) \in (\mathbb{R}^2 \setminus \{(0,0)\})$ may be functions of $\frac{m}{m_H}$:

$$\rho^4 t_{\rho^{-1}\mathbf{m}}^{\tilde{u}u}(\rho y) - t_{\mathbf{m}}^{\tilde{u}u}(y) = -\hbar^2 (\rho^4 t_{\text{fish}}^{\rho^{-1}M}(\rho y) - t_{\text{fish}}^M(y)) = 0. \quad (107)$$

With that, $t_{\mathbf{m}}^{\tilde{u}u}$ does not contribute to the RG-flow: $c_{2\rho}^{(1)} = 0$.

Remark 6.1. When using the renormalization method (B), we have to weaken a bit the sm-expansion axiom given in [Düt15]. In detail: among other conditions, this axiom requires that the term $l = 0$ in the sm-expansion⁹

$$t_{\mathbf{m}}(y) = \sum_{l=0}^L \sum_{l_1, l_2 \geq 0, l_1 + l_2 = l} m^{2l_1} m_H^{2l_2} u_{l_1, l_2}^{(\mathbf{m})}(y) + \mathfrak{r}_{2L+2}^{(\mathbf{m})}(y),$$

i.e. $u_{0,0}^{(\mathbf{m})}$, is independent of \mathbf{m} . Only the distributions $u_{l_1, l_2}^{(\mathbf{m})}$ with $l_1 + l_2 \geq 1$ may be polynomials in $(\log \frac{m}{M_1}, \log \frac{m_H}{M_1})$, where $M_1 > 0$ is a fixed mass scale. From (104)-(105) we explicitly see that this condition is violated by the method (B); e.g. for $M := m$ we have

$$\square_y \left(\frac{\log(-m^2(y^2 - i0))}{y^2 - i0} \right) = \square_y \left(\frac{\log(-M_1^2(y^2 - i0))}{y^2 - i0} \right) + 8i\pi \delta(y) \log \frac{m}{M_1}.$$

So, when using method (A), we keep the original version of the sm-expansion axiom; but, when using method (B), we admit that also $u_{0,0}^{(\mathbf{m})}$ is a polynomial in $(\log \frac{m}{M_1}, \log \frac{m_H}{M_1})$. Proceeding analogously to [Düt15], one verifies that using method (B) and the inductive Epstein-Glaser construction of time-ordered products, this weakened version of the sm-expansion axiom can be fulfilled to all orders of perturbation theory.

⁹In 4 dimensions only *even* powers of m and m_H appear.

Conjecture: *If we renormalize all t -distributions in all inductive steps of the Epstein-Glaser construction by the choice (B), i.e. we use as renormalization mass scale throughout $M := \alpha_1 m + \alpha_2 m_H$ (where (α_1, α_2) are as above, different values of (α_1, α_2) for different diagrams are admitted), then the RG-flow is trivial:*

$$z_\rho(L) = L/\hbar \quad \forall \rho > 0. \quad (108)$$

Proof. We prove this Conjecture for massless, primitive divergent diagrams.¹⁰ This covers all massive 1-loop diagrams with singular order $\omega = 0$ or 1, because for these diagrams, only the leading term of the sm-expansion, which is the corresponding massless distribution, contributes to the RG-flow. However, note that also the $(\omega = 2)$ -diagrams (181)-(182) are covered, because their scaling behaviour can be traced back to the scaling behaviour of the massless fish-diagram, see Appendix A.

Let $y := (y_1, \dots, y_l)$, $Y_j := y_j^2 - i0$; for the considered diagrams the unrenormalized distribution $t^\circ \in \mathcal{D}'(\mathbb{R}^{4l} \setminus \{0\})$ scales homogeneously:

$$\rho^{\omega+4l} t^\circ(\rho y) = t^\circ(y). \quad (109)$$

We work with an analytic regularization [Hol08]:

$$t^{\zeta^\circ}(y) := t^\circ(y) (M^{2l} Y_1 \dots Y_l)^\zeta, \quad (110)$$

where $\zeta \in \mathbb{C} \setminus \{0\}$ with $|\zeta|$ sufficiently small, and $M > 0$ is a renormalization mass scale. t^{ζ° scales also homogeneously – by the regularization we gain that the degree (of the scaling) is $(\omega + 4l - 2l\zeta)$, which is not an integer. Therefore, the homogeneous extension $t^\zeta \in \mathcal{D}'(\mathbb{R}^{4l})$ is unique and can be obtained by differential renormalization [DFKR14, Sect. IV.D]:

$$t^\zeta(y) = \frac{1}{\prod_{j=0}^{\omega} (j - \omega + 2l\zeta)} \sum_{r_1 \dots r_{\omega+1}} \partial_{y_{r_{\omega+1}}} \dots \partial_{y_{r_1}} \left(\overline{y_{r_1} \dots y_{r_{\omega+1}} t^{\zeta^\circ}(y)} \right), \quad (111)$$

where $\sum_r \partial_{y_r}(y_r \dots) := \sum_r \partial_\mu^{y_r}(y_r^\mu \dots)$ and the overline denotes the direct extension. In order that the limit $\zeta \rightarrow 0$ exists, we subtract from the Laurent series t^ζ its principle part. According to [DFKR14, Corollary 4.4] the term $\sim \zeta^0$ (“minimal subtraction”) is an admissible extension t^M of t° :

$$t^M(y) = \frac{(-1)^\omega}{\omega!} \sum_{r_1 \dots r_{\omega+1}} \partial_{y_{r_{\omega+1}}} \dots \partial_{y_{r_1}} \left[\frac{1}{2l} \left(\overline{y_{r_1} \dots y_{r_{\omega+1}} t^\circ(y) \log(M^{2l} Y_1 \dots Y_l)} \right) + \left(\sum_{j=1}^{\omega} \frac{1}{j} \right) \left(\overline{y_{r_1} \dots y_{r_{\omega+1}} t^\circ(y)} \right) \right], \quad (112)$$

¹⁰That is, massless diagrams Γ with singular order $\omega(\Gamma) \geq 0$ (see footnote 8) which do not contain any subdiagram $\Gamma_1 \subset \Gamma$ with less vertices and with $\omega(\Gamma_1) \geq 0$. For example, the setting sun diagram is a primitive divergent 2-loop diagram.

see [DFKR14, formula (104)]. The second term is of the form $\sum_{|a|=\omega} C_a \partial^a \delta(y)$. The first term breaks homogeneous scaling in y logarithmically, but we explicitly see that

$$\rho^{\omega+4l} t^{\rho^{-1}M}(\rho y) = t^M(y) ; \quad (113)$$

this proves the Conjecture. \square

Remark 6.2. We only admit renormalizations of the initial $U(1)$ -Higgs model which fulfill PGI. This requirement is neither in conflict with method (A) nor with method (B), for the following reason: we require PGI only for the initial model, i.e. only at one fixed scale. Now, working at one fixed scale, the renormalization constant M appearing in (105) may have any value $M > 0$ for both methods (A) and (B) and, hence, one may choose it such that PGI is satisfied. These methods only prescribe how M behaves under a scaling transformation: using (A) it remains unchanged, using (B) it is also scaled: $M \mapsto \rho^{-1}M$.

Computation of $b_{0\rho}^{(1)}$: The purpose of this computation is to illustrate the methods (A) and (B) for a 1-loop coefficient having contributions from more than one Feynman diagram; in addition this computation is also a preparation for the following Subsection.

To compute $b_{0\rho}^{(1)}$ we have to take the following terms of $T_{2\mathbf{m}}(L_1(x_1) \otimes L_1(x_2))$ into account:

$$t_{\mathbf{m}\lambda\nu}^{\partial B\partial B}(x_1 - x_2) \partial^\lambda B(x_1) \partial^\nu B(x_2) + \left(t_{\mathbf{m}\nu}^{B\partial B}(x_1 - x_2) B(x_1) \partial^\nu B(x_2) + (x_1 \leftrightarrow x_2) \right), \quad (114)$$

where $(x_1 \leftrightarrow x_2)$ refers only to the $t^{B\partial B}$ -term and

$$\begin{aligned} t_{\mathbf{m}\lambda\nu}^{\partial B\partial B}(x_1 - x_2) &:= \omega_0 \left(T_{2\mathbf{m}}(\varphi A_\lambda(x_1) \otimes \varphi A_\nu(x_2)) \right), \\ t_{\mathbf{m}\nu}^{B\partial B}(x_1 - x_2) &:= -\omega_0 \left(T_{2\mathbf{m}}(\partial^\lambda \varphi A_\lambda(x_1) \otimes \varphi A_\nu(x_2)) \right). \end{aligned} \quad (115)$$

The unrenormalized t -distributions read

$$\begin{aligned} t_{\mathbf{m}\lambda\nu}^{\partial B\partial B^\circ}(y) &= -\hbar^2 g_{\lambda\nu} \Delta_m^F(y) \Delta_{m_H}^F(y) \in \mathcal{D}'(\mathbb{R}^4 \setminus \{0\}), \\ t_{\mathbf{m}\nu}^{B\partial B^\circ}(y) &= \hbar^2 \Delta_m^F(y) \partial_\nu \Delta_{m_H}^F(y) \in \mathcal{D}'(\mathbb{R}^4 \setminus \{0\}); \end{aligned} \quad (116)$$

both scale homogeneously, e.g. $\rho^5 t_{\rho^{-1}\mathbf{m}\nu}^{B\partial B^\circ}(\rho y) = t_{\mathbf{m}\nu}^{B\partial B^\circ}(y)$.

We renormalize both diagrams by using method (A). Since $t^{\partial B\partial B^\circ}$ essentially agrees with $t^{\tilde{u}u^\circ}$, we know from (106) that

$$\rho^4 t_{\rho^{-1}\mathbf{m}\lambda\nu}^{\partial B\partial B}(\rho y) - t_{\mathbf{m}\lambda\nu}^{\partial B\partial B}(y) = -\hbar^2 g_{\lambda\nu} C_{\text{fish}} \log \rho \delta(y). \quad (117)$$

To extend $t_{\mathbf{m}\nu}^{B\partial B^\circ}$, we use again the sm-expansion

$$t_{\mathbf{m}\nu}^{B\partial B^\circ}(y) = v_\nu^\circ(y) + r_{\mathbf{m}\nu}^\circ(y), \quad v_\nu^\circ(y) := \hbar^2 D^F(y) \partial_\nu D^F(y) = \frac{\hbar^2}{2} \partial_\nu t_{\text{fish}}^\circ(y), \quad (118)$$

the statements in the preceding example about the remainder $r_{\mathbf{m}\nu}^\circ$ and its extension $r_{\mathbf{m}\nu} \in \mathcal{D}'(\mathbb{R}^4)$ hold true also in the present case, with the exception that now $\text{sd}(r_{\mathbf{m}\nu}^\circ) = 3$. All extensions of v_ν° with a minimal (i.e. logarithmic) breaking of homogeneous scaling in y can be obtained by differential renormalization:

$$v_\nu^M(y) = \frac{\hbar^2}{2} \partial_\nu t_{\text{fish}}^M(y) \in \mathcal{D}'(\mathbb{R}^4), \quad M > 0 \text{ arbitrary.} \quad (119)$$

Choosing M according to method (A) we get

$$\rho^5 t_{\rho^{-1}\mathbf{m}\nu}^{B\partial B^\circ}(\rho y) - t_{\mathbf{m}\nu}^{B\partial B^\circ}(y) = \rho^5 v_\nu^M(\rho y) - v_\nu^M(y) = \frac{\hbar^2}{2} C_{\text{fish}} \log \rho \partial_\nu \delta(y). \quad (120)$$

Taking (96) into account we see that the terms (114) give

$$\begin{aligned} Z_\rho^{(2)}(L(g)^{\otimes 2}) &= C_{\text{fish}} \kappa^2 \hbar^2 \log \rho \int dx_1 dx_2 g(x_1)g(x_2) \\ &\cdot \left(\partial_\nu \delta(x_1 - x_2) B(x_1) \partial^\nu B(x_2) - g_{\lambda\nu} \delta(x_1 - x_2) \partial^\lambda B(x_1) \partial^\nu B(x_2) + \dots \right) + \dots, \end{aligned} \quad (121)$$

which yields

$$z_\rho(L) = \hbar^{-1} \left(L - \frac{i C_{\text{fish}} \hbar \kappa^2}{2} \log \rho (1+1) (\partial B)^2 + \dots \right) \quad (122)$$

by using (95) and (82). We end up with

$$b_{0\rho}^{(1)} = -2i C_{\text{fish}} \log \rho = \frac{-1}{4\pi^2} \log \rho. \quad (123)$$

The conjecture can explicitly be verified: renormalizing $t^{B\partial B^\circ}$ or $t^{\partial B\partial B^\circ}$ by means of method (B) the pertinent expressions (117) and (120), respectively, vanish. Hence, also the values $b_{0\rho}^{(1)} = \frac{-1}{8\pi^2} \log \rho$ and $b_{0\rho}^{(1)} = 0$ can appear.

Note that

$$\omega_0 \left(T_{2\mathbf{m}}(A\partial\varphi(x_1) \otimes A\partial\varphi(x_2)) \right) B(x_1)B(x_2)$$

contributes only to $b_{1\rho}^{(1)}$ and not to $b_{0\rho}^{(1)}$, see (212) and (217).

6.2 Equality of certain coefficients

In this subsection we explain how some of the equations (36)-(46) (which express the geometrical interpretability at all scales) can be fulfilled on 1-loop level, by renormalizing such that certain Feynman diagrams, which go over into each other by exchanging $B \leftrightarrow \varphi$ for some lines, give the same contribution to the RG-flow (up to possibly different combinatorial factors).

How to obtain $c_{0\rho}^{(1)} = b_{0\rho}^{(1)}$: The terms contributing to $c_{0\rho}^{(1)}$ are obtained from (114)-(115) by exchanging $B \leftrightarrow \varphi$ throughout. The corresponding unrenormalized distributions

$t^{\partial\varphi\partial\varphi^\circ}$ and $t^{\varphi\partial\varphi^\circ}$ are given by (116) with $\Delta_{m_H}^F$ replaced by Δ_m^F . However, this modification does not show up in the pertinent massless parts $-g_{\lambda\nu}t_{\text{fish}}^\circ$ (104) and v_ν° (118), respectively. Since only the latter contribute to the RG-flow, we conclude that

$$c_{0\rho}^{(1)} = b_{0\rho}^{(1)} \quad (124)$$

can be obtained in the following way:

- (\star) *Corresponding t -distributions (or more precisely their massless parts) have to be renormalized all with method (A) or all with method (B).* For the various t -distributions we may choose different renormalization mass scales M when using method (A); or different linear combinations $M = \alpha_1 m + \alpha_2 m_H$ when using method (B).

Taking Remark 6.2 into account, we see that this renormalization prescription is compatible with PGI of the initial $U(1)$ -Higgs model.

Having obtained (124), the equations (39), (41) and (43)-(44) simplify to

$$l_{3\rho}^{(1)} = l_{4\rho}^{(1)}, \quad l_{5\rho}^{(1)} = l_{6\rho}^{(1)}, \quad l_{7\rho}^{(1)} = l_{8\rho}^{(1)} = l_{9\rho}^{(1)} \quad (125)$$

on 1-loop level.

Obtaining analogously $l_{1\rho}^{(1)} = l_{2\rho}^{(1)}$ (37): There are contributions to $l_{1\rho}^{(1)}$ coming from $T_2(L_1 \otimes L_2)$, more precisely only the part $L_1^1 := BA\partial\varphi - \varphi A\partial B$ of L_1 contributes. These terms read

$$\begin{aligned} & 2\left(\omega_0\left(T_2(A_\lambda B(x_1) \otimes A_\nu B(x_2))\right)\partial^\lambda\varphi(x_1)A^\nu(x_2)B(x_2)\right. \\ & \quad \left.- \omega_0\left(T_2(A\partial B(x_1) \otimes A_\nu B(x_2))\right)\varphi(x_1)A^\nu(x_2)B(x_2)\right) + (x_1 \leftrightarrow x_2). \end{aligned} \quad (126)$$

The corresponding contributions to $l_{2\rho}^{(1)}$ are obtained by exchanging $B \leftrightarrow \varphi$ throughout. Proceeding similarly to the derivation of (124) (in particular the renormalization prescription (\star) is used), we find that the contributions of these terms to $l_{1\rho}^{(1)}$ and $l_{2\rho}^{(1)}$ agree.

Note that similarly to (214)-(215), there is neither a contribution to $l_{1\rho}^{(1)}$ nor to $l_{2\rho}^{(1)}$ coming from the following $T_2(L_1^1 \otimes L_2)$ -term:

$$-\frac{m_H^2}{m^2}\omega_0\left(T_2\left((B\partial_\lambda\varphi - \varphi\partial_\lambda B)(x_1) \otimes B\varphi(x_2)\right)\right)A^\lambda(x_1)B(x_2)\varphi(x_2) + (x_1 \leftrightarrow x_2). \quad (127)$$

The contributions to $l_{1\rho}^{(1)}, l_{2\rho}^{(1)}$ coming from $T_3(L_1^{\otimes 3})$ use only the part L_1^1 of L_1 , the relevant terms of $T_3(L_1^{\otimes 3})$ are triangle diagrams with 2 or 3 derivatives, they are of the form

$$\begin{aligned} & \left((v_{11}^{\lambda\nu}(y_1, y_2) + r_{11}^{\lambda\nu}(y_1, y_2)) A_\lambda(x_1)\partial_\nu\varphi(x_2)B(x_3) \right. \\ & \quad - (v_{12}^{\lambda\nu}(y_1, y_2) + r_{12}^{\lambda\nu}(y_1, y_2)) A_\lambda(x_1)\partial_\nu B(x_2)\varphi(x_3) \\ & \quad + (v_{21}^\lambda(y_1, y_2) + r_{21}^\lambda(y_1, y_2)) A_\lambda(x_3)B(x_1)\varphi(x_2) \\ & \quad \left. - (v_{22}^\lambda(y_1, y_2) + r_{22}^\lambda(y_1, y_2)) A_\lambda(x_3)\varphi(x_1)B(x_2) \right) + \left(5 \text{ permutations of } x_1, x_2, x_3 \right), \end{aligned} \quad (128)$$

where $y_j := x_j - x_3$ and the first two lines are obtained from each other by exchanging $B \leftrightarrow \varphi$ throughout and the same for the third and fourth lines. Moreover, we have inserted the sm-expansion. The remainders r_{kl} ($k, l \in \{1, 2\}$) do not contribute to the RG-flow, since they are renormalized by direct extension. The unrenormalized versions of the massless parts v_{kl} agree pairwise: $v_k^\circ := v_{k1}^\circ = v_{k2}^\circ \in \mathcal{D}'(\mathbb{R}^8 \setminus \{0\})$; explicitly they read

$$\begin{aligned} v_1^{\lambda\nu\circ}(y_1, y_2) &= \hbar^3 \left(-\partial^\nu D^F(y_1) D^F(y_2) \partial^\lambda D^F(y_1 - y_2) + \partial^\lambda \partial^\nu D^F(y_1) D^F(y_2) D^F(y_1 - y_2) \right), \\ v_2^{\lambda\circ}(y_1, y_2) &= -2\hbar^3 \partial^\lambda \partial_\nu D^F(y_1) \partial^\nu D^F(y_2) D^F(y_1 - y_2). \end{aligned} \quad (129)$$

Obviously these v° -distributions scale homogeneously in (y_1, y_2) . Renormalization breaks this symmetry by terms of the form

$$\begin{aligned} \rho^8 v_{1l}^{\lambda\nu}(\rho y_1, \rho y_2) - v_{1l}^{\lambda\nu}(y_1, y_2) &= \hbar^3 \log \rho C_1 g^{\lambda\nu} \delta(y_1, y_2), \\ \rho^9 v_{2l}^\lambda(\rho y_1, \rho y_2) - v_{2l}^\lambda(y_1, y_2) &= \hbar^3 \log \rho (C_{21} \partial_{y_1}^\lambda + C_{22} \partial_{y_2}^\lambda) \delta(y_1, y_2), \end{aligned} \quad (130)$$

where Lorentz covariance is taken into account.¹¹ According to the prescription (\star) we have to choose the renormalization mass scales for v_{k1} and v_{k2} by the *same* method.

Inserting these results into

$$Z_\rho^{(3)}(L_1^{\mathbf{m}}(g)^{\otimes 3}) = \sigma_\rho \circ T_{3\rho^{-1}\mathbf{m}}(\sigma_\rho^{-1}(L_1^{\mathbf{m}}(g))^{\otimes 3}) - T_{3\mathbf{m}}(L_1^{\mathbf{m}}(g)^{\otimes 3}) + \dots, \quad (131)$$

where $L_1^{\mathbf{m}}(g) := \int dx L_1^{\mathbf{m}}(x) g(x)$, we obtain the following contributions to $l_{1\rho}^{(1)}$ and $l_{2\rho}^{(1)}$, respectively: using method (A) throughout, we get

$$l_{1\rho}^{(1)} = (-C_1 - C_{21} + C_{22} - 3i C_{\text{fish}}) \log \rho = l_{2\rho}^{(1)}, \quad (132)$$

where the C_{fish} -term is the contribution from (126). When using (B) for v_1° or v_2° (or for both), the constant C_1 or $(-C_{21} + C_{22})$, resp., (or both) is/are replaced by zero, and analogously for the contribution from (126). In all these cases $l_{1\rho}^{(1)} = l_{2\rho}^{(1)}$ remains true.

Obtaining analogously $l_{5\rho}^{(1)} = l_{6\rho}^{(1)}$ and $l_{7\rho}^{(1)} = l_{9\rho}^{(1)}$ (125): the terms contributing to $l_{5\rho}^{(1)}$ and $l_{7\rho}^{(1)}$ are listed in Appendix B. The corresponding terms contributing to $l_{6\rho}^{(1)}$ and $l_{9\rho}^{(1)}$, respectively, are obtained by replacing $B \leftrightarrow \varphi$ throughout. Proceeding as above, the renormalization prescription (\star) implies $l_{5\rho}^{(1)} = l_{6\rho}^{(1)}$ and $l_{7\rho}^{(1)} = l_{9\rho}^{(1)}$.

Obtaining analogously $l_{3\rho}^{(1)} = l_{4\rho}^{(1)}$ and $l_{7\rho}^{(1)} = l_{8\rho}^{(1)}$ (125): here, the combinatorics is somewhat involved – there is not a (1 : 1)-correspondence of terms. In Appendix B these two equations are verified by explicit computation of the pertinent coefficients, under the assumption that all contributing terms are renormalized by method (A). From the calculations given there, we see that $l_{3\rho}^{(1)} = l_{4\rho}^{(1)}$ and $l_{7\rho}^{(1)} = l_{8\rho}^{(1)}$ hold true, also if the method (B) is used for corresponding terms. For example, if we switch to method (B) in (226) and (231), $C_{1\Delta}$ is replaced by zero in (228) and (233), but $l_{3\rho}^{(1)} = l_{4\rho}^{(1)}$ remains true.

¹¹In terms of the invariants $C_{j\Delta}$ computed in Appendix A, we have $C_1 = -C_{1\Delta} + C_{2\Delta} = -2C_{1\Delta}$. The computation of the invariants C_{21} and C_{22} is a more difficult task – for our purposes, we do not need to know these numbers.

6.3 Vanishing of the A^4 -term due to maintenance of PC

In this short subsection we explain, why the identity (45) holds true to 1-loop order.

A byproduct of the calculations in Appendix C is the following (see Remark C.2): working out stability of PC under the RG-flow,

$$\lim_{\varepsilon \downarrow 0} [Q, S(i z_\rho(L)(g_\varepsilon))]_\star \approx 0, \quad (133)$$

to order κ^4 , we obtain – among other relations – the equation

$$0 \approx l_{11\rho}^{(1)} \lim_{\varepsilon \downarrow 0} \int dx (g_\varepsilon(x))^2 [Q, (A^2)^2(x)] = l_{11\rho}^{(1)} 4i \lim_{\varepsilon \downarrow 0} \int dx (g_\varepsilon(x))^2 A^2 A \partial u(x). \quad (134)$$

Using results of Appendix A of [DS00] we may argue as follows: since there does not exist a local field polynomial W^μ such that $A^2 A \partial u = \partial_\mu W^\mu$, the equation (134) implies

$$l_{11\rho}^{(1)} = 0. \quad (135)$$

6.4 Changing the running interaction by finite renormalizations

To continue the fulfillment of the identities (36)-(46) on 1-loop level, we take into account that the following finite renormalizations are admitted by the axioms of causal perturbation theory [EG73, DF04] and that they preserve PGI of the initial model: to $T_2(L_1(x_1) \otimes L_1(x_2))$ we may add

$$\begin{aligned} & \hbar^2 \delta(x_1 - x_2) \log \frac{m}{M} \left(\alpha_1 (\partial\varphi)^2(x_1) + \alpha_2 m_H^2 \varphi^2(x_1) + \alpha_3 F^2(x_1) + \alpha_4 (\partial A + mB)^2 \right. \\ & \quad + \alpha_5 (-m^2 B^2(x_1) + (\partial B)^2(x_1)) + \alpha_6 (m^2 A^2(x_1) - (\partial A)^2(x_1)) \\ & \quad \left. + \alpha_7 m^2 (-2\tilde{u}u(x_1) + A^2(x_1) - B^2(x_1)) \right), \end{aligned} \quad (136)$$

where $\alpha_1, \dots, \alpha_7 \in \mathbb{C}$ are arbitrary.

The compatibility with PGI is obvious for the α_1 -, α_2 -, α_3 - and α_4 -term, because the commutator of Q with the pertinent field polynomials is ≈ 0 . For the other terms, the PGI-relation

$$[Q, T_2(L_1(x_1) \otimes L_1(x_2))]_\star \approx i \partial_\nu^{x_1} T_2(P_1^\nu(x_1) \otimes L_1(x_2)) + (x_1 \leftrightarrow x_2) \quad (137)$$

is maintained, if we simultaneously renormalize $T_2(P_1^\nu(x_1) \otimes L_1(x_2))$ by adding

$$\hbar^2 \delta(x_1 - x_2) \log \frac{m}{M} \left(\alpha_5 2m u \partial^\nu B(x_1) + (\alpha_6 + \alpha_7) 2m^2 A^\nu u(x_1) \right). \quad (138)$$

Proceeding analogously to the computation (95)-(101) of $c_{2\rho}^{(1)}$, we find that the renor-

malizations (136) modify the 1-loop coefficients $e_\rho^{(1)}$ appearing in $z_\rho(L)$ (19) as follows:

$$a_{0\rho}^{(1)} \mapsto a_{0\rho}^{(1)} + 2i \alpha_3 \log \rho , \quad (139)$$

$$a_{1\rho}^{(1)} \mapsto a_{1\rho}^{(1)} - i (\alpha_6 + \alpha_7) \log \rho , \quad (140)$$

$$a_{2\rho}^{(1)} \mapsto a_{2\rho}^{(1)} + i (\alpha_4 - \alpha_6) \log \rho , \quad (141)$$

$$b_{0\rho}^{(1)} \mapsto b_{0\rho}^{(1)} - i \alpha_5 \log \rho , \quad (142)$$

$$b_{1\rho}^{(1)} \mapsto b_{1\rho}^{(1)} + i (\alpha_4 - \alpha_5 - \alpha_7) \log \rho , \quad (143)$$

$$b_{2\rho}^{(1)} \mapsto b_{2\rho}^{(1)} + i \alpha_4 \log \rho , \quad (144)$$

$$c_{0\rho}^{(1)} \mapsto c_{0\rho}^{(1)} - i \alpha_1 \log \rho , \quad (145)$$

$$c_{1\rho}^{(1)} \mapsto c_{1\rho}^{(1)} + i \alpha_2 \log \rho , \quad (146)$$

$$c_{2\rho}^{(1)} \mapsto c_{2\rho}^{(1)} - i \alpha_7 \log \rho , \quad (147)$$

the other coefficients remain unchanged.

Remark 6.3. There are further, linearly independent (w.r.t. \simeq) possibilities for finite renormalization which preserves PGI:

- to $T_2(L_1(x_1) \otimes L_1(x_2))$ we may add

$$\hbar^2 \delta(x_1 - x_2) \log \frac{m}{M} \beta_1 (2 \partial \tilde{u} \partial u(x_1) - (\partial A)^2(x_1) + (\partial B)^2(x_1)) , \quad (148)$$

since $[Q, (2 \partial \tilde{u} \partial u - (\partial A)^2 + (\partial B)^2)]_\star \approx -2i \partial^\mu (\partial A \partial_\mu u)$;

- to $T_2(L_2(x_1) \otimes L_1(x_2))$ we may add

$$\begin{aligned} & \hbar^2 \delta(x_1 - x_2) \log \frac{m}{M} \left(\beta_2 \frac{m_H^2}{2m} \varphi^3(x_1) \right. \\ & \left. + \beta_3 \left[m A^2 \varphi - m \tilde{u} u \varphi + B(A \partial \varphi) - \varphi(A \partial B) - \frac{m_H^2}{2m} \varphi^3 - \frac{m_H^2}{2m} B^2 \varphi \right](x_1) \right) , \end{aligned} \quad (149)$$

since $[\dots] = L_1$ and $[Q, L_1]_\star \approx i \partial_\nu P_1^\nu$ (P_1^ν is given in (70));

- to $T_2(L_2(x_1) \otimes L_2(x_2))$ we may add

$$\hbar^2 \delta(x_1 - x_2) \log \frac{m}{M} \beta_4 \frac{m_H^2}{4m^2} \varphi^4(x_1) . \quad (150)$$

However, the β_1 - and β_3 -renormalization add “by hand” novel kind of terms $\sim \partial \tilde{u} \partial u$ and $\sim m \tilde{u} u \varphi$, respectively, to $(z_\rho(L) - L)$ (19) – therefore, we do not take them into account. And, even if we would admit a $\partial \tilde{u} \partial u$ - and a $(m \tilde{u} u \varphi)$ -term in $(z_\rho(L) - L)$, the β_1 - and β_3 -renormalization cannot be used to fulfill the crucial identities (155) or (151), because they do not change $a_{2\rho}^{(1)} - b_{0\rho}^{(1)}$ or $l_{3\rho}^{(1)} - l_{0\rho}^{(1)}$, respectively.

We may not use the β_2 - and β_4 -renormalization: they would destroy the relations $l_{3\rho}^{(1)} = l_{4\rho}^{(1)}$ and $l_{7\rho}^{(1)} = l_{8\rho}^{(1)} = l_{9\rho}^{(1)}$ since they would modify only $l_{3\rho}^{(1)}$ and $l_{7\rho}^{(1)}$, respectively.

6.5 Geometrical interpretation at all scales

There are two necessary conditions for the geometrical interpretation at all scales, which are crucial, since they cannot be fulfilled by finite renormalizations.

Verification of the first crucial necessary condition: The condition (47) reads to 1-loop level

$$l_{7\rho}^{(1)} - l_{3\rho}^{(1)} = l_{5\rho}^{(1)} - l_{0\rho}^{(1)}. \quad (151)$$

As discussed in Remark 6.3, there is no possibility to fulfill this equation by finite renormalizations. Therefore, we investigate its validity by explicit calculation: using the renormalization method (A) for all contributing terms, the results of Appendix B yield:

$$\frac{l_{7\rho}^{(1)} - l_{3\rho}^{(1)}}{\log \rho} = 4 C_{1\Delta} + \frac{m^2}{m_H^2} 8 (iC_{2\Box} - C_{2\Delta}), \quad (152)$$

$$\frac{l_{5\rho}^{(1)} - l_{0\rho}^{(1)}}{\log \rho} = 8 C_{1\Delta} - 4i C_{1\Box}; \quad (153)$$

where cancellations of fish- with triangle-, fish- with square- and triangle- with square-diagrams are not used so far. Using now relations among the invariants $C_{j\Delta}$ and $C_{j\Box}$ (derived in Appendix A), we find that (151) holds indeed true:

$$l_{7\rho}^{(1)} - l_{3\rho}^{(1)} = 4 C_{1\Delta} = l_{5\rho}^{(1)} - l_{0\rho}^{(1)}. \quad (154)$$

The fact that we need cancellations of square- and triangle-contributions shows that (151) is of a deeper kind than the equalities derived in Sect. 6.2.

The identity (151) holds also if certain terms are renormalized by method (B), e.g. all contributing triangle and square-diagrams with (B) and all contributing fish diagrams with (A), or vice versa.

A further example, for which both sides of (151) vanish, is given below under the subtitle ‘‘How to fulfill BRST-invariance of the running Lagrangian’’.

How to fulfill the second crucial necessary condition: the condition (46) reads to 1-loop order

$$b_{2\rho}^{(1)} = \frac{1}{2} (a_{2\rho}^{(1)} + b_{1\rho}^{(1)} - a_{1\rho}^{(1)} - b_{0\rho}^{(1)}). \quad (155)$$

Performing the finite renormalizations (136), i.e. inserting (139)-(147) into (155), we find that all α_j drop out – that is, the condition (155) cannot be fulfilled by means of these finite renormalizations.

Inserting the explicit values (123), (216) and (217) for the coefficients $a_{j\rho}^{(1)}$, $b_{j\rho}^{(1)}$, computed by using method (A), we obtain

$$\begin{aligned} & \frac{1}{\log \rho} \left(\frac{1}{2} (a_{2\rho}^{(1)} + b_{1\rho}^{(1)} - a_{1\rho}^{(1)} - b_{0\rho}^{(1)}) - b_{2\rho}^{(1)} \right) \\ &= iC_{\text{fish}} \left(\left(2 - \frac{1}{4} + 1 - 3 \right) + \frac{m_H^2}{m^2} \left(\frac{1}{2} - \frac{1}{4} \right) + \frac{m_H^4}{m^4} \left(-\frac{1}{2} \right) \right). \end{aligned} \quad (156)$$

Hence, using method (A) throughout, we have $\lambda_{12\rho} \neq 0$, i.e. the geometrical interpretation is violated by terms $\sim A\partial B$.

To fulfill the condition (155), we may proceed as follows: we use

- method (B) for the terms (211)-(212) [i.e. $b_{1\rho}^{(1)}$] and (208) [i.e. $a_{0\rho}^{(1)}$ and part of $a_{1\rho}^{(1)}$];
- and method (A) for (115) [i.e. $b_{0\rho}^{(1)}$], (210) [i.e. part of $a_{1\rho}^{(1)}$] and (213) [i.e. $b_{2\rho}^{(1)}$].

With that the values (216)-(217) are modified:

$$a_{0\rho}^{(1)} = 0, \quad a_{1\rho}^{(1)} = -4i C_{\text{fish}} \log \rho \quad \text{and} \quad b_{1\rho}^{(1)} = 0, \quad (157)$$

and $a_{2\rho}^{(1)}, b_{0\rho}^{(1)}, b_{2\rho}^{(1)}$ remain unchanged.

Fulfilling the remaining conditions by finite renormalizations: to complete the fulfillment of the identities (36)-(46) to 1-loop order, we show that we can reach by finite renormalizations that the numbers D_1, D_2, D_3 , defined by

$$D_1 \log \rho := l_{1\rho}^{(1)} - l_{0\rho}^{(1)} - \frac{1}{2}(b_{0\rho}^{(1)} - a_{1\rho}^{(1)}), \quad (158)$$

$$D_2 \log \rho := l_{3\rho}^{(1)} - l_{0\rho}^{(1)} - (c_{1\rho}^{(1)} - a_{1\rho}^{(1)}), \quad (159)$$

$$D_3 \log \rho := l_{5\rho}^{(1)} - 2l_{0\rho}^{(1)} + a_{1\rho}^{(1)}, \quad (160)$$

vanish. For the coefficients $e_\rho^{(1)}$ appearing in these definitions we use values which fulfill the equations (37), (124), (125), (135), (151) and (155).

If $c_{1\rho}^{(1)}, l_{0\rho}^{(1)}, l_{3\rho}^{(1)}$ and $l_{5\rho}^{(1)}$ are renormalized by method (A) (see Appendix B) and $a_{1\rho}^{(1)}$ as in (157), we have $D_3 = 0$ and $D_2 = 0$.¹² However, to be as general as possible, we admit arbitrary values of D_1, D_2, D_3 in the following.

Using (139)-(147), we see that we have to solve the following system of linear equations:

$$\begin{aligned} D_1 + \frac{i}{2}(\alpha_5 - (\alpha_6 + \alpha_7)) &= 0 \\ D_2 - i(\alpha_2 + (\alpha_6 + \alpha_7)) &= 0 \\ D_3 - i(\alpha_6 + \alpha_7) &= 0. \end{aligned} \quad (161)$$

There is a unique solution for $(\alpha_2, \alpha_5, (\alpha_6 + \alpha_7))$. To preserve $b_{0\rho}^{(1)} = c_{0\rho}^{(1)}$ we have to choose $\alpha_1 = \alpha_5$ (145). There remains a 3-dimensional freedom of renormalization: α_3, α_4 and $(\alpha_6 - \alpha_7)$ are unrestricted.

As a summary we explicitly give a particular solution for the coefficients $e_\rho^{(1)}$, which fulfills the geometrical interpretation at all scales: using the method (B) only for the terms specified before (157) (in order that we have the values (157)) and renormalizing all other terms with method (A), and then performing the α_5 -renormalization with $\alpha_5 = 2i D_1 = 2i l_1 - 4 C_{\text{fish}}$ (161) and the pertinent α_1 -renormalization with $\alpha_1 = \alpha_5$, we end

¹²Since we have not computed $l_{1\rho}^{(1)}$ (see footnote 11), we cannot make a corresponding statement about the value of D_1 .

up with:

$$\begin{aligned}
a_{0\rho}^{(1)} &= a_{2\rho}^{(1)} = l_{11\rho}^{(1)} = 0, & a_{1\rho}^{(1)} &= -4i C_{\text{fish}} \log \rho, \\
b_{0\rho}^{(1)} &= c_{0\rho}^{(1)} = (2i C_{\text{fish}} + 2l_1) \log \rho, & b_{1\rho}^{(1)} &= (4i C_{\text{fish}} + 2l_1) \log \rho, \\
b_{2\rho}^{(1)} &= 3i C_{\text{fish}} \log \rho, & c_{1\rho}^{(1)} &= -i \left(6 \frac{m^2}{m_H^2} + 5 \frac{m_H^2}{m^2} \right) C_{\text{fish}} \log \rho, & c_{2\rho}^{(1)} &= -i C_{\text{fish}} \log \rho, \\
l_{0\rho}^{(1)} &= -3i C_{\text{fish}} \log \rho, & l_{1\rho}^{(1)} &= l_{2\rho}^{(1)} =: l_1 \log \rho, \\
l_{3\rho}^{(1)} &= l_{4\rho}^{(1)} = i \left(1 - 6 \frac{m^2}{m_H^2} - 5 \frac{m_H^2}{m^2} \right) C_{\text{fish}} \log \rho, & l_{5\rho}^{(1)} &= l_{6\rho}^{(1)} = -2i C_{\text{fish}} \log \rho, \\
l_{7\rho}^{(1)} &= l_{8\rho}^{(1)} = l_{9\rho}^{(1)} = i \left(2 - 6 \frac{m^2}{m_H^2} - 5 \frac{m_H^2}{m^2} \right) C_{\text{fish}} \log \rho, & & & & (162)
\end{aligned}$$

where l_1 is the number which one obtains on computing $l_{1\rho}^{(1)} =: l_1 \log \rho$ by method (A) – from (132) and footnote 11 we have

$$l_1 = -3i C_{\text{fish}} + 2 C_{1\Delta} - C_{21} + C_{22} = -\frac{5i}{2} C_{\text{fish}} - C_{21} + C_{22}. \quad (163)$$

How to fulfill BRST-invariance of the running Lagrangian (54): we start with the values (162), except that we do not perform the finite renormalization with $\alpha_1 = \alpha_5 = -4 C_{\text{fish}} + 2i l_1$, with that we have $b_{1\rho}^{(1)} = 0$ and $b_{0\rho}^{(1)} = c_{0\rho}^{(1)} = -2i C_{\text{fish}}$.

To fulfill the BRST-condition

$$b_{\rho}^{(1)} := b_{0\rho}^{(1)} = c_{0\rho}^{(1)} = a_{1\rho}^{(1)} = -b_{2\rho}^{(1)} = l_{0\rho}^{(1)} = l_{1\rho}^{(1)} = l_{2\rho}^{(1)} = l_{5\rho}^{(1)} = l_{6\rho}^{(1)} \quad (164)$$

(see (59)-(60)), there is the trivial possibility $b_{\rho}^{(1)} = 0$, which is obtained by renormalizing all contributing terms by method (B). However, there is also the solution $b_{\rho}^{(1)} = -3i C_{\text{fish}} \log \rho$ which can be obtained from our starting values as follows: we perform finite renormalizations with $\alpha_1 = \alpha_5 = C_{\text{fish}}$ and $\alpha_7 = -C_{\text{fish}}$; this yields $b_{0\rho}^{(1)} = c_{0\rho}^{(1)} = a_{1\rho}^{(1)} = -b_{2\rho}^{(1)} = l_{0\rho}^{(1)} = -3i C_{\text{fish}} \log \rho$ and $c_{2\rho}^{(1)} = 0$ (61), and does not change $c_{1\rho}^{(1)}$ and $b_{1\rho}^{(1)} = 0 = a_{2\rho}^{(1)}$. In order that $l_{1\rho}^{(1)} = l_{2\rho}^{(1)}$ and $l_{5\rho}^{(1)} = l_{6\rho}^{(1)}$ also get the value $-3i C_{\text{fish}} \log \rho$ we switch the method from (A) to (B) in the triangle terms of $l_{1\rho}^{(1)} = l_{2\rho}^{(1)}$ (i.e. in (128)) and in the term(s) (236) (or alternatively (237) and (239)) of $l_{5\rho}^{(1)} = l_{6\rho}^{(1)}$.

The condition $l_{\rho}^{(1)} := c_{1\rho}^{(1)} = l_{3\rho}^{(1)} = l_{4\rho}^{(1)} = l_{7\rho}^{(1)} = l_{8\rho}^{(1)} = l_{9\rho}^{(1)}$ (60) can non-trivially be satisfied by replacing the terms $\mathcal{O}((\frac{m^2}{m_H^2})^0)$ by zero in the expressions (162) for $l_{3\rho}^{(1)} = l_{4\rho}^{(1)}$ and $l_{7\rho}^{(1)} = l_{8\rho}^{(1)} = l_{9\rho}^{(1)}$; that is, we switch the method from (A) to (B) in the terms (226), (231), (243) and (248).

Taking into account that $a_{0\rho}^{(1)}$ is not restricted by BRST-invariance (i.e. the finite renormalization parameter α_3 can freely be chosen), we get the following particular so-

lution for the parameters (55):

$$\frac{a_{0\rho}^{(1)}}{\log \rho} \in \mathbb{C} \text{ arbitrary, } b_\rho^{(1)} = -3i C_{\text{fish}} \log \rho, \quad l_\rho^{(1)} = -i \left(6 \frac{m^2}{m_H^2} + 5 \frac{m_H^2}{m^2} \right) C_{\text{fish}} \log \rho. \quad (165)$$

Remark 6.4. We discuss whether there is a non-trivial renormalization of the gauge-fixing parameter to 1-loop order (28):

- if we fulfill the geometrical interpretation as described (i.e. (157) and (161) are satisfied) and choose $\alpha_3 = 0$ and $\alpha_4 = \alpha_6$, we have $a_{0\rho}^{(1)} = 0$ and $a_{2\rho}^{(1)} = 0$ which yields

$$\Lambda_\rho = 1 + \mathcal{O}(\hbar^2 \kappa^4). \quad (166)$$

- In contrast, if we use the renormalization method (A) throughout and do not perform any finite renormalization, the values (216) give

$$\Lambda_\rho = 1 - \frac{1}{24\pi^2} \log \rho \hbar \kappa^2 + \mathcal{O}(\hbar^2 \kappa^4). \quad (167)$$

However, we recall that even BRST-invariance of $L_0 + z_\rho(L)$ (54) does not restrict $a_{0\rho}^{(1)}$ in any way; hence, we are free to modify $a_{0\rho}^{(1)}$ by a finite renormalization (139) and this changes Λ_ρ to 1-loop order.

7 PGI for tree diagrams for the running interaction

Besides the geometrical interpretability at all scales and BRST-invariance, there is a further property which we will investigate for the running Lagrangian: PGI-tree. Its restrictive power for a general renormalizable ansatz for the interaction and the importance of that are pointed out in the Introduction. In [Düt05] it is generally proved that BRST-invariance of the Lagrangian (that is (54) in our case) implies PGI-tree. For interactions which are only tri- and quadrilinear in the fields, it has turned out that PGI-tree restricts the interaction as strong as BRST-invariance of the Lagrangian; however, we will see that for $z_\rho(L)$, which contains also bilinear terms, PGI-tree is much less restrictive.

Definition of PGI-tree: to study PGI-tree, it suffices to consider the *connected* tree diagrams. To select the latter from the S -functionals appearing in the PGI-condition (68), we first introduce the connected time-ordered products $(T_n^c)_{n \in \mathbb{N}}$, by the (usual) recursive definition

$$T_n^c(F_1 \otimes \dots \otimes F_n) := T_n(F_1 \otimes \dots \otimes F_n) - \sum_{|P| \geq 2} \prod_{J \in P} T_{|J|}^c(F_{j_1} \otimes \dots \otimes F_{j_{|J|}}), \quad (168)$$

where $\{j_1, \dots, j_{|J|}\} = J$, $j_1 < \dots < j_{|J|}$, the sum runs over all partitions P of $\{1, \dots, n\}$ in at least two subsets and \prod means the classical product. Analogously to (16), let S^c be

the generating functional of the connected time-ordered products. PGI (68) is equivalent to PGI for S^c , i.e.

$$[Q, S^c(i\mathcal{L}(g))]_{\star} \approx \frac{d}{d\eta}|_{\eta=0} S^c(i\mathcal{L}(g) + \eta\mathcal{P}^\nu(g; \partial_\nu g)) ; \quad (169)$$

this can be verified straightforwardly by using that $[Q, \cdot]_{\star}$ is a graded derivation w.r.t. classical product, see [Düt05, Lemma 1].

For a connected time ordered product $T_n^c(\mathcal{L}(g)^{\otimes n})$, the tree diagrams are the terms of lowest order in \hbar , if the interaction $\mathcal{L}(g)$ is homogeneous in \hbar , see e.g. [DF01]. If, as usual, $\mathcal{L}(g) \sim \hbar^{-1}$ and $\mathcal{P}(g; \partial g) \sim \hbar^0$, the tree diagrams of $S^c(i\mathcal{L}(g))$ [or $\frac{d}{d\eta}|_{\eta=0} S^c(i z_\rho(L)(g) + \eta\mathcal{P}(g; \partial g))$] are precisely the terms $\sim \hbar^{-1}$ [or $\sim \hbar^0$, resp.], and all connected loop diagrams are of higher orders in \hbar . Taking into account that $[Q, F]_{\star} \sim \hbar$ if $F \sim \hbar^0$ (see again [Düt05, Lemma 1]), we define: *PGI-tree is the equation (169) to lowest order in \hbar , which is \hbar^0 .*

But $z_\rho(L)$ is by itself a formal power series in \hbar . Therefore, we use a trick to select the tree diagrams from $S^c(i z_\rho(L)(g))$ and $\frac{d}{d\eta}|_{\eta=0} S^c(i z_\rho(L)(g) + \eta\mathcal{P}^\nu(g; \partial_\nu g))$. Namely, *in all coefficients e_ρ (20) (and nowhere else) we replace \hbar by another parameter τ* ; however, in particular the factors \hbar^{-1} for each vertex (see (19)) and \hbar for each propagator remain untouched. Note that this substitution concerns also the pertinent Q -vertex: \hbar is replaced by τ in (73) and in (75). With that, we have $z_\rho(L) \sim \hbar^{-1}$ and $\mathcal{P}(g; \partial g) \sim \hbar^0$, and we can apply the above given definition of PGI-tree to $S(i z_\rho(L)(g))$. After the selection of the tree diagrams we reset $\tau := \hbar$.

Remark 7.1. Writing the interaction $\mathcal{L}(g) = z_\rho(L)(g)$ and the pertinent Q -vertex $\mathcal{P}(g; f)$ by means of the τ -trick, the proof in [Düt05] that BRST-invariance of the Lagrangian implies PGI-tree applies to $L_0 + z_\rho(L)$ (54). In addition this proof yields an explicit expression for the Q -vertex [Düt05, formula (3.23)], which gives

$$P_0^{(1)\nu} = 0, \quad P_1^{(1)\nu} = b_\rho^{(1)} (m A^\nu u \varphi - \partial^\nu B u \varphi + B u \partial^\nu \varphi), \quad P_2^{(1)\nu} = b_\rho^{(1)} (A^\nu u \varphi^2 + A^\nu u B^2), \quad (170)$$

if (54) holds true. ($b_\rho^{(1)}$ is defined by (59).)

Restrictions on the 1-loop coefficients of $z_\rho(L)$ coming from PGI-tree: here we assume that the coefficients e_ρ of $z_\rho(L)$ are unknown. In Appendix C it is worked out that PGI-tree for $\mathcal{L}(g) = z_\rho(L)(g)$ can be fulfilled to order τ^1 iff the following relations

among the 1-loop coefficients $e_\rho^{(1)}$ hold true:

$$\begin{aligned}
& a_{0\rho}^{(1)}, b_{0\rho}^{(1)}, b_{1\rho}^{(1)}, c_{0\rho}^{(1)}, c_{2\rho}^{(1)}, l_{7\rho}^{(1)}, l_{1\rho}^{(1)} \text{ are arbitrary,} \\
& a_{2\rho}^{(1)} = 0, \quad l_{11\rho}^{(1)} = 0, \quad a_{1\rho}^{(1)} = b_{0\rho}^{(1)} + 2c_{2\rho}^{(1)} - b_{1\rho}^{(1)}, \quad -b_{2\rho}^{(1)} = l_{0\rho}^{(1)} = b_{0\rho}^{(1)} + c_{2\rho}^{(1)} - b_{1\rho}^{(1)}, \\
& l_{2\rho}^{(1)} = l_{1\rho}^{(1)}, \quad l_{3\rho}^{(1)} = l_{7\rho}^{(1)} + c_{2\rho}^{(1)} + 2(b_{0\rho}^{(1)} - \frac{b_{1\rho}^{(1)}}{2} - l_{1\rho}^{(1)}), \\
& l_{4\rho}^{(1)} = l_{7\rho}^{(1)} + c_{2\rho}^{(1)} + 3b_{0\rho}^{(1)} - c_{0\rho}^{(1)} - b_{1\rho}^{(1)} - 2l_{1\rho}^{(1)} = l_{3\rho}^{(1)} + (b_{0\rho}^{(1)} - c_{0\rho}^{(1)}), \\
& l_{5\rho}^{(1)} = 2l_{1\rho}^{(1)} - b_{0\rho}^{(1)}, \quad l_{6\rho}^{(1)} = 2l_{1\rho}^{(1)} - c_{0\rho}^{(1)} = l_{5\rho}^{(1)} + (b_{0\rho}^{(1)} - c_{0\rho}^{(1)}), \\
& l_{8\rho}^{(1)} = l_{7\rho}^{(1)} + (b_{0\rho}^{(1)} - c_{0\rho}^{(1)}), \quad l_{9\rho}^{(1)} = l_{7\rho}^{(1)} + 2(b_{0\rho}^{(1)} - c_{0\rho}^{(1)}), \\
& c_{1\rho}^{(1)} = l_{7\rho}^{(1)} + 2c_{2\rho}^{(1)} + 4(b_{0\rho}^{(1)} - \frac{b_{1\rho}^{(1)}}{2} - l_{1\rho}^{(1)}) + \frac{2m^2}{m_H^2}(l_{1\rho}^{(1)} - b_{0\rho}^{(1)} + \frac{b_{1\rho}^{(1)}}{2}). \tag{171}
\end{aligned}$$

Let us compare these PGI-tree relations with the geometrical interpretability at all scales on 1-loop level (i.e. equations (36)-(46) to first order in $\hbar\kappa^2$): from the number of free parameters (7 versus 9) we immediately see that the geometrical interpretability cannot imply PGI-tree. Also the reversed statement does not hold true: in order that (171) implies the geometrical interpretability, precisely one additional relation is needed, namely

$$l_{1\rho}^{(1)} = b_{0\rho}^{(1)} - \frac{b_{1\rho}^{(1)}}{2}. \tag{172}$$

However, note that (171) implies the two crucial necessary conditions for the geometrical interpretability, (155) and (151), *without* this additional relation (172). Note also that the geometrical interpretability does not imply (172).

One verifies straightforwardly, that the particular solution (162) for the 1-loop coefficients $e_\rho^{(1)}$, generalized by an arbitrary finite renormalization of $a_{0\rho}^{(1)}$ (139), solves the system of linear equations (171)-(172), *i.e. there exists a way to renormalize such that PGI-tree and the geometrical interpretability are satisfied*. In contrast to the latter, the system (171)-(172) fixes the values of the finite renormalization parameters α_3 and $(\alpha_6 - \alpha_7)$ uniquely (cf. the discussion after (161)), this reflects that (171)-(172) is more restrictive.

Relation to minimal subtraction: dimensional regularization with minimal subtraction is a widespread scheme in conventional momentum space renormalization, which preserves BRST-invariance generically. Applied to the 1-loop diagrams of our initial model, this property implies that the resulting time-ordered products fulfill PGI.¹³ In the minimal subtraction scheme the mass scale(s) is/are chosen in a way which belongs to the class “use always method (A) and do not perform any finite renormalization”. Using the latter prescription, neither PGI-tree nor the geometrical interpretability are maintained under the RG-flow, because the second crucial necessary condition (155) is

¹³We are not aware of a proof of this statement, but it is very plausible. A corresponding statement for higher loop diagrams involves a partial adiabatic limit, because such diagrams contain inner vertices, which are integrated out with $g(x) = 1$ in conventional momentum space renormalization – but PGI is formulated *before* the adiabatic limit $g \rightarrow 1$ is taken.

violated. Weakening this prescription by admitting the finite PGI-preserving renormalizations (139)-(147), the violation of (155) cannot be removed.¹⁴

Remark 7.2. For an ansatz for the interaction containing solely trilinear and quadrilinear terms it has been worked out for various models that PGI-tree determines the interaction essentially uniquely¹⁵ (see e.g. [Sto97, DS99, ADS99, Sch01, DGBSV10]). But here, for an ansatz containing also bilinear terms, we obtain crucially different results:¹⁶

- BRST-invariance of the total Lagrangian does not determine the interaction uniquely (see (56));
- PGI-tree is truly weaker than BRST-invariance of the Lagrangian. (Compared with (56), the relations (171) leave 4 additional parameters to be freely chosen.)

This can be understood as follows: PGI presupposes that the free theory is BRST-invariant: $s_0 L_0 \simeq 0$. If we try to trace back the case of an interaction including bilinear terms to the case with solely tri- and quadrilinear terms, by renormalization of the wave functions and parameters (27)-(31), BRST-invariance of the free theory may get lost. Explicitly we obtain¹⁷

$$s_0 L_0^o \simeq 0 \quad \Leftrightarrow \quad b_{0\rho} = 0 = a_{2\rho} \wedge a_{1\rho} = b_{1\rho} = c_{2\rho} . \quad (173)$$

To 1-loop order we can simultaneously fulfil this condition and BRST-invariance of the Lagrangian (54): by using the renormalization method (B) for the relevant diagrams, we can reach that in the particular solution (165) of (54) the value for $b_\rho^{(1)}$ is replaced by 0. But in general (173) does not hold true, see e.g. the particular solution (162) of the geometrical interpretability. Moreover, there is the additional obstacle that, after the renormalization of the wave functions and parameters, the interaction still contains the bilinear term $b_{2\rho} m A \partial B$.

- In [DS00] it is worked out for the model of three massive vector fields that, making a general renormalizable ansatz for the interaction, the condition of PC for tree diagrams (PC-tree) restricts the interaction to the same extent as PGI-tree – the essentially unique solution is the $SU(2)$ -Higgs-Kibble model. However, for our $S(iz_\rho(L)(g))$, which contains also bilinear terms, PC-tree is significantly weaker than PGI-tree. This follows from our results: we have proved that PC (and, hence, also PC-tree) holds true, but in general PGI-tree is violated.

¹⁴An alternative, simple argument that PGI-tree (and, hence, also PGI) can get lost under the RG-flow is the following: The α_1 -renormalization (145) maintains PGI of the initial model, but it can be used to violate the PGI-tree equations (171), since it modifies only $c_{0\rho}^{(1)}$ – this argumentation works also for the α_2 -renormalization (146).

¹⁵This holds also for our model. Namely, setting $a_{j\rho}^{(1)} = 0$, $b_{j\rho}^{(1)} = 0$ and $c_{j\rho}^{(1)} = 0$ (for all j), the restrictions from PGI-tree (171) and (289) yield $l_{k\rho}^{(1)} = 0$ (for all k).

¹⁶We are not aware of any other paper in which PGI has been studied for an interaction containing bilinear terms.

¹⁷Since $L_0^o = L_0 + z_\rho(L)_{\text{bilinear}}$, where $z_\rho(L)_{\text{bilinear}}$ is the bilinear part of $z_\rho(L)$ (19) *without* the $A\partial B$ -term, the easiest way to obtain the equivalence (173) is to work out the condition $s_0 z_\rho(L)_{\text{bilinear}} \simeq 0$.

8 Summary and concluding remarks

Defining the RG-flow by means of a scaling transformation [HW03, DF04, BDF09] one can easily show that PC is maintained under the RG-flow. Hence, the $U(1)$ -Higgs model is a consistent QFT-model at all scales. However, the somewhat stronger property of PGI gets lost in general, and in particular if one uses a renormalization prescription corresponding to minimal subtraction.

Using the Epstein-Glaser axioms [EG73, DF04], completed by the requirement that the initial model fulfills PGI, the RG-flow contains quite a large non-uniqueness, due to the following two facts:

- whether a certain Feynman diagram contributes to the RG-flow, depends on whether one chooses as renormalization mass scale a fixed mass (method (A)), or a mass which is subject to our scaling transformation – e.g. the mass of one of the basic fields (method (B)).
- By finite renormalizations (136), which preserve PGI of the initial model, one can modify the RG-flow.

To 1-loop level we have shown that, by using this non-uniqueness, one can achieve that the geometrical interpretation is possible at all scales; one can even achieve that the much stronger condition of BRST-invariance of the running Lagrangian is satisfied. But this requires a quite (geometrical interpretation) or very (BRST-invariance) specific prescription for the choice of the renormalization method ((A) or (B)) for the various Feynman diagrams, and for the finite renormalizations. If one uses always method (A) – minimal subtraction is of this kind – the geometrical interpretation is violated by terms $\sim A\partial B$; relaxing this prescription by admitting finite PGI-preserving renormalizations, these $A\partial B$ -terms cannot be removed.

Instead of a state independent renormalization scheme, as e.g. minimal subtraction, one may use state dependent renormalization conditions: e.g. in the adiabatic limit the vacuum expectation values of certain time ordered products must agree with the “experimentally” known values for the masses of stable particles in the vacuum, and analogous conditions for parameters of certain vacuum correlation functions. With such a scheme, quite a lot of diagrams are renormalized by method (A). To 1-loop level, the geometrical interpretability at all scales amounts then mainly to the question, whether it is nevertheless possible to fulfill the second crucial necessary condition (155), which requires to renormalize certain diagrams by method (B), see (156)-(157). We postpone this question to future work, and we do so also for the dependence of our results on the initial value of the gauge-fixing parameter.

Returning to the fundamental question, already touched in the Introduction, whether masses are really generated by the Higgs mechanism, we may say that our results sow a germ of doubt.

Or – one can keep the Higgs mechanism as a fundamental principle explaining the origin of mass at all scales (although it is not understood in a pure QFT framework),

then our results forbid quite a lot of renormalization schemes, in particular minimal subtraction!

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A Breaking of homogeneous scaling for some 1-loop diagrams

In Sect. 6.1 it is derived that the violation of homogeneous scaling of the massless fish diagram is

$$(y\partial_y + 4) t_{\text{fish}}^M(y) = C_{\text{fish}} \delta(y) \quad \text{with} \quad C_{\text{fish}} = \frac{-i}{8\pi^2}, \quad y\partial_y := y_\lambda \partial_y^\lambda, \quad (174)$$

and that for the fish diagram t_{m,m_H}^M (102) (with different masses m, m_H) it holds

$$(y\partial_y + 4 - m\partial_m - m_H\partial_{m_H}) t_{m,m_H}^M(y) = C_{\text{fish}} \delta(y), \quad (175)$$

where $M > 0$ is a renormalization mass scale.

In this appendix we compute the breaking of homogeneous scaling for some massless triangle diagrams,

$$t_{1\Delta}^{\mu\nu\circ}(y) := \partial^\mu D^F(y_1) \partial^\nu D^F(y_2) D^F(y_1 - y_2) \in \mathcal{D}'(\mathbb{R}^8 \setminus \{0\}), \quad (176)$$

$$t_{2\Delta}^{\mu\nu\circ}(y) := D^F(y_1) D^F(y_2) \partial^\mu \partial^\nu D^F(y_1 - y_2) \in \mathcal{D}'(\mathbb{R}^8 \setminus \{0\}), \quad (177)$$

some massless square diagrams,

$$t_{1\Box}^{\lambda\nu\circ}(y) := \partial^\lambda \partial^\nu D^F(y_1 - y_2) \partial_\mu D^F(y_2 - y_3) D^F(y_3) \partial^\mu D^F(y_1) \in \mathcal{D}'(\mathbb{R}^{12} \setminus \{0\}), \quad (178)$$

$$t_{2\Box}^{\lambda\nu\circ}(y) := D^F(y_1 - y_2) \partial^\nu \partial_\mu D^F(y_2 - y_3) D^F(y_3) \partial^\lambda \partial^\mu D^F(y_1) \in \mathcal{D}'(\mathbb{R}^{12} \setminus \{0\}), \quad (179)$$

$$t_{3\Box}^{\lambda\nu\circ}(y) := \partial^\lambda D^F(y_1 - y_2) \partial^\nu \partial_\mu D^F(y_2 - y_3) D^F(y_3) \partial^\mu D^F(y_1) \in \mathcal{D}'(\mathbb{R}^{12} \setminus \{0\}), \quad (180)$$

and for some massive fish-like diagrams,

$$t_{1m,m_H}^{\mu\nu\circ}(y) := \Delta_m^F(y) \partial^\mu \partial^\nu \Delta_{m_H}^F(y) \in \mathcal{D}'(\mathbb{R}^4 \setminus \{0\}), \quad (181)$$

$$t_{2m,m_H}^{\mu\nu\circ}(y) := \partial^\mu \Delta_m^F(y) \partial^\nu \Delta_{m_H}^F(y) \in \mathcal{D}'(\mathbb{R}^4 \setminus \{0\}), \quad (182)$$

by using the renormalization method (A) (see Sect. 6.1). The point is that these computations can be traced back to the result (174).

Massless triangle diagrams: first note that contraction of $t_{2\Delta}^{\mu\nu\circ}$ with $g_{\mu\nu}$ yields

$$t_{2\Delta\mu}^{\mu\circ} = -i \delta(y_1 - y_2) t_{\text{fish}}^\circ(y_1) \quad (183)$$

by using $\square D^F(x) = -i\delta(x)$. Hence, for an arbitrary pair of almost homogeneous extensions to $\mathcal{D}'(\mathbb{R}^8)$, the difference is of the form

$$t_{2\Delta\mu}^\mu(y) + i \delta(y_1 - y_2) t_{\text{fish}}(y_1) = C \delta(y), \quad C \in \mathbb{C}; \quad (184)$$

such a term scales homogeneously. We conclude that

$$(y\partial_y + 8) t_{2\Delta\mu}^\mu(y) = -i C_{\text{fish}} \delta(y), \quad \text{where} \quad y\partial_y := y_1^\mu \partial_\mu^{y_1} + y_2^\mu \partial_\mu^{y_2}. \quad (185)$$

Due to Lorentz covariance, the expression $(y\partial_y + 8) t_{2\Delta}^{\mu\nu}(y)$ must be $\sim g^{\mu\nu}$; therefore, we obtain

$$\rho^8 t_{2\Delta}^{\mu\nu}(\rho y) - t_{2\Delta}^{\mu\nu}(y) = C_{2\Delta} g^{\mu\nu} \delta(y) \log \rho \quad \text{with} \quad C_{2\Delta} = \frac{-i}{4} C_{\text{fish}}. \quad (186)$$

To compute the violation of homogeneous scaling for $t_{1\Delta}^{\mu\nu}$, we introduce

$$\tilde{t}_\Delta^\mu(y) := \partial^\mu D^F(y_1) D^F(y_2) D^F(y_1 - y_2), \quad (187)$$

which exists in $\mathcal{D}'(\mathbb{R}^8)$ by the direct extension (see footnote 8) and scales homogeneously: $(y\partial_y + 7) \tilde{t}_\Delta^\mu(y) = 0$. In $\mathcal{D}'(\mathbb{R}^8 \setminus \{0\})$ we find

$$(\partial_{y_1}^\nu + \partial_{y_2}^\nu) \tilde{t}_\Delta^{\mu\circ}(y) = t_{2\Delta}^{\mu\nu\circ}(y_1 - y_2, -y_2) + t_{1\Delta}^{\mu\nu\circ}(y). \quad (188)$$

Therefore, arbitrary almost homogeneous extensions fulfill

$$(\partial_{y_1}^\nu + \partial_{y_2}^\nu) \tilde{t}_\Delta^\mu(y) = t_{2\Delta}^{\mu\nu}(y_1 - y_2, -y_2) + t_{1\Delta}^{\mu\nu}(y) + \tilde{C} \delta(y) \quad (189)$$

for some $\tilde{C} \in \mathbb{C}$. We conclude

$$0 = (y\partial_y + 8) (\partial_{y_1}^\nu + \partial_{y_2}^\nu) \tilde{t}_\Delta^\mu(y) = (y\partial_y + 8) t_{2\Delta}^{\mu\nu}(y_1 - y_2, -y_2) + (y\partial_y + 8) t_{1\Delta}^{\mu\nu}(y). \quad (190)$$

Taking (186) into account we end up with

$$\rho^8 t_{1\Delta}^{\mu\nu}(\rho y) - t_{1\Delta}^{\mu\nu}(y) = C_{1\Delta} g^{\mu\nu} \delta(y) \log \rho \quad \text{with} \quad C_{1\Delta} = -C_{2\Delta} = \frac{i}{4} C_{\text{fish}}. \quad (191)$$

Massless square diagrams: proceeding analogously, we use that

$$g_{\lambda\nu} t_{1\Box}^{\lambda\nu\circ}(y) = -i \delta(y_1 - y_2) t_{1\Delta\mu}^{\mu\circ}(y_1, y_1 - y_3) \quad (192)$$

and obtain

$$\rho^{12} t_{1\Box}^{\lambda\nu}(\rho y) - t_{1\Box}^{\lambda\nu}(y) = C_{1\Box} g^{\lambda\nu} \delta(y) \log \rho \quad \text{with} \quad C_{1\Box} = -i C_{1\Delta} = \frac{1}{4} C_{\text{fish}} . \quad (193)$$

Taking into account that in $\mathcal{D}'(\mathbb{R}^{12} \setminus \{0\})$ it holds

$$\partial_{y_2}^\nu \left(\partial^\lambda D^F(y_1 - y_2) \partial_\mu D^F(y_2 - y_3) D^F(y_3) \partial^\mu D^F(y_1) \right) = -t_{1\Box}^{\lambda\nu\circ}(y) + t_{3\Box}^{\lambda\nu\circ}(y) , \quad (194)$$

we conclude that

$$\rho^{12} t_{3\Box}^{\lambda\nu}(\rho y) - t_{3\Box}^{\lambda\nu}(y) = C_{3\Box} g^{\lambda\nu} \delta(y) \log \rho \quad \text{with} \quad C_{3\Box} = C_{1\Box} . \quad (195)$$

Finally, by means of

$$\partial_{y_1}^\lambda \left(D^F(y_1 - y_2) \partial^\nu \partial_\mu D^F(y_2 - y_3) D^F(y_3) \partial^\mu D^F(y_1) \right) = t_{3\Box}^{\lambda\nu\circ}(y) + t_{2\Box}^{\lambda\nu\circ}(y) , \quad (196)$$

we derive that

$$\rho^{12} t_{2\Box}^{\lambda\nu}(\rho y) - t_{2\Box}^{\lambda\nu}(y) = C_{2\Box} g^{\lambda\nu} \delta(y) \log \rho \quad \text{with} \quad C_{2\Box} = -C_{3\Box} = -C_{1\Box} . \quad (197)$$

Similarly to the massless fish diagram (106), the following holds also for the massless triangle diagrams (176)-(177) and for the massless square diagrams (178)-(180): the breaking of homogeneous scaling is *equal* for all almost homogeneous extensions. This must be so, because two almost homogeneous extensions differ by a term of the form $\sum_{|a|=\omega} C_a \partial^a \delta$, $C_a \in \mathbb{C}$, which scales homogeneously. (See footnote 8 for the definition of ω ; for the examples in hand we have $\omega = 0$.)

Massive fish-like diagrams: as a preparation we first compute the violation of homogeneous scaling of the renormalized version $t_2^{\mu\nu M}$ of the massless distribution

$$t_2^{\mu\nu\circ}(y) := \partial^\mu D^F(y) \partial^\nu D^F(y) \in \mathcal{D}'(\mathbb{R}^4 \setminus \{0\}) . \quad (198)$$

This computation can be traced back to the result (174) in the following way: first we write $t_2^{\mu\nu\circ}$ as

$$t_2^{\mu\nu\circ}(y) = \frac{y^\mu y^\nu}{48} \square_y \square_y t_{\text{fish}}^\circ(y) \in \mathcal{D}'(\mathbb{R}^4 \setminus \{0\}) , \quad (199)$$

which follows from the explicit formula $D^F(y) = \frac{-1}{4\pi^2 (y^2 - i0)}$ by straightforward calculation, taking into account that “ $y \neq 0$ ”. Then, by differential renormalization we get

$$t_2^{\mu\nu M}(y) = \frac{y^\mu y^\nu}{48} \square_y \square_y t_{\text{fish}}^M(y) \in \mathcal{D}'(\mathbb{R}^4) , \quad (200)$$

where $M > 0$ is a fixed mass scale (method (A)). From this relation we conclude that

$$(y \partial_y + 6) t_2^{\mu\nu M}(y) = \frac{C_{\text{fish}}}{48} y^\mu y^\nu \square_y \square_y \delta(y) = \frac{C_{\text{fish}}}{12} (g^{\mu\nu} \square_y + 2 \partial_y^\mu \partial_y^\nu) \delta(y) ; \quad (201)$$

the second equality is obtained by straightforward calculation.

We renormalize the massive fish-like diagrams (181), (182) by using the sm-expansion [Düt15]. Due to that we know that the violation of homogeneous scaling is of the form

$$(y\partial_y + 6 - m\partial_m - m_H\partial_{m_H}) t_{j,m,m_H}^{\mu\nu M}(y) = \left(C_{j1} g^{\mu\nu} \square_y + C_{j2} \partial_y^\mu \partial_y^\nu \right) \delta(y) + g^{\mu\nu} \left(m^2 P_{j1}(\log \frac{m}{M}, \log \frac{m_H}{M}) + m_H^2 P_{j2}(\log \frac{m}{M}, \log \frac{m_H}{M}) \right) \delta(y), \quad j = 1, 2, \quad (202)$$

where $P_{jl}(z_1, z_2)$ is a polynomial in z_1 and z_2 . The term $\mathcal{O}(m^0)$ can be computed by setting $m := 0 =: m_H$; hence, we know the values of the numbers $(C_{2l})_{l=1,2}$ from (201). We renormalize such that the relations

$$\begin{aligned} \partial_y^\mu \partial_y^\nu t_{m,m_H}^\circ(y) &= t_{1m,m_H}^{\mu\nu \circ}(y) + t_{1m_H,m}^{\mu\nu \circ}(y) + t_{2m,m_H}^{\mu\nu \circ}(y) + t_{2m_H,m}^{\mu\nu \circ}(y), \\ g_{\mu\nu} t_{1m,m_H}^{\mu\nu \circ}(y) &= -m_H^2 t_{m,m_H}^\circ(y) \quad \text{and} \quad t_{2m,m_H}^{\mu\nu \circ}(y) = t_{2m_H,m}^{\nu\mu \circ}(y) \end{aligned} \quad (203)$$

are maintained up to (local) terms which are in the kernel of the operator $(y\partial_y + 6 - m\partial_m - m_H\partial_{m_H})$; for the first and the last relation this is a term of the form

$$(C_1 g^{\mu\nu} \square_y + C_2 \partial_y^\mu \partial_y^\nu) \delta(y) + g^{\mu\nu} (m^2 C_3 + m_H^2 C_4) \delta(y), \quad C_k \in \mathbb{C} \text{ arbitrary.}$$

Due to the sm-expansion, this renormalization prescription restricts only the local terms $\mathcal{O}(m^2, m_H^2)$: without this prescription the numbers C_3 and C_4 may be replaced by polynomials in $\log \frac{m}{M}$ and $\log \frac{m_H}{M}$.

By using the renormalized version of the relations (203) and (175) and (201), we determine the numbers C_{1l} and the polynomials P_{jl} . It results

$$\begin{aligned} \rho^6 t_{1m/\rho, m_H/\rho}^{\mu\nu M}(\rho y) - t_{1m, m_H}^{\mu\nu M}(y) \\ = C_{\text{fish}} \left[\frac{1}{12} \left(-g^{\mu\nu} \square_y + 4 \partial_y^\mu \partial_y^\nu \right) - \frac{g^{\mu\nu}}{4} m_H^2 \right] \delta(y) \log \rho, \end{aligned} \quad (204)$$

$$\begin{aligned} \rho^6 t_{2m/\rho, m_H/\rho}^{\mu\nu M}(\rho y) - t_{2m, m_H}^{\mu\nu M}(y) \\ = C_{\text{fish}} \left[\frac{1}{12} \left(g^{\mu\nu} \square_y + 2 \partial_y^\mu \partial_y^\nu \right) + \frac{g^{\mu\nu}}{8} (m^2 + m_H^2) \right] \delta(y) \log \rho. \end{aligned} \quad (205)$$

B Computation of some 1-loop coefficients of the running interaction

In this appendix we compute some 1-loop coefficients e of $z_\rho(L)$, defined by (19), (20) and

$$e \log \rho := e_\rho^{(1)}, \quad (206)$$

by using the results of Appendix A. We assume that for all contributing terms the renormalization mass scale M is chosen according to method (A), see Sect. 6.1. Working

in Feynman gauge, we may use the following conventions:

$$\begin{aligned}
\omega_0\left(T_2(B(x) \otimes B(y))\right) &= \hbar \Delta_m^F(x-y), & \omega_0\left(T_2(\varphi(x) \otimes \varphi(y))\right) &= \hbar \Delta_{m_H}^F(x-y), \\
\omega_0\left(T_2(u(x) \otimes \tilde{u}(y))\right) &= \hbar \Delta_m^F(x-y), & \omega_0\left(T_2(\tilde{u}(x) \otimes u(y))\right) &= -\hbar \Delta_m^F(x-y), \\
\omega_0\left(T_2(A^\mu(x) \otimes A^\nu(y))\right) &= -\hbar g^{\mu\nu} \Delta_m^F(x-y), & (\square + m^2)\Delta_m^F(x) &= -i\delta(x). \quad (207)
\end{aligned}$$

Coefficients of some bilinear fields: to compute a_0, a_1, a_2, b_1, b_2 , we have to take into account the following terms of $T_2(L_1^{2\otimes 2})$ where $L_1^2 := m A^2\varphi + BA\partial\varphi - \varphi A\partial B - \frac{m_H^2}{2m} B^2\varphi$: the most complicated is

$$\begin{aligned}
&\omega_0\left(T_2((B\partial^\mu\varphi - \varphi\partial^\mu B)(x_1) \otimes (B\partial^\nu\varphi - \varphi\partial^\nu B)(x_2))\right) A_\mu(x_1)A_\nu(x_2) \\
&= \left(-t_{1m,m_H}^{\mu\nu M}(y) - t_{1m_H,m}^{\mu\nu M}(y) + t_{2m,m_H}^{\mu\nu M}(y) + t_{2m_H,m}^{\mu\nu M}(y)\right) A_\mu(x_1)A_\nu(x_2), \quad (208)
\end{aligned}$$

where $y := x_1 - x_2$. Using (204)-(205), we find that (208) gives the following contribution to $Z_\rho^{(2)}(L(g)^{\otimes 2})$ (96):

$$\begin{aligned}
&\kappa^2 \hbar^2 C_{\text{fish}} \int dx_1 dx_2 g(x_1)g(x_2) \left(\frac{1}{3}(g^{\mu\nu}\square_y - \partial_y^\mu\partial_y^\nu) + \frac{1}{2}g^{\mu\nu}(m^2 + m_H^2)\right) \delta(y) A_\mu(x_1)A_\nu(x_2) \\
&= \kappa^2 \hbar^2 C_{\text{fish}} \int dx (g(x))^2 \left(\frac{-1}{6}F^2(x) + \frac{1}{2}(m^2 + m_H^2)A^2(x)\right) + \dots, \quad (209)
\end{aligned}$$

where the dots stand for terms with derivatives of g , which do not contribute to the adiabatic limit. The further contributing terms are

$$m^2 4\omega_0\left(T_2(A^\mu\varphi(x_1) \otimes A^\nu\varphi(x_2))\right) A_\mu(x_1)A_\nu(x_2), \quad (210)$$

$$\frac{m_H^4}{4m^2} 4\omega_0\left(T_2(B\varphi(x_1) \otimes B\varphi(x_2))\right) B(x_1)B(x_2), \quad (211)$$

$$\omega_0\left(T_2(A\partial\varphi(x_1) \otimes A\partial\varphi(x_2))\right) B(x_1)B(x_2) = g_{\mu\nu} t_{1m,m_H}^{\mu\nu M}(y) B(x_1)B(x_2), \quad (212)$$

$$\begin{aligned}
&-m 2\omega_0\left(T_2(A^\mu\varphi(x_1) \otimes A^\nu\varphi(x_2))\right) A_\mu(x_1)\partial_\nu B(x_2) + (x_1 \leftrightarrow x_2) \\
&+ m 2\omega_0\left(T_2(A^\mu\varphi(x_1) \otimes A\partial\varphi(x_2))\right) A_\mu(x_1)B(x_2) + (x_1 \leftrightarrow x_2), \quad (213)
\end{aligned}$$

$$- \frac{m_H^2}{2m} 2\omega_0\left(T_2((B\partial^\mu\varphi - \varphi\partial^\mu B)(x_1) \otimes B\varphi(x_2))\right) A_\mu(x_1)B(x_2) + (x_1 \leftrightarrow x_2). \quad (214)$$

The last term does not contribute to the RG-flow, because in the sm-expansion of the pertinent unrenormalized expression the leading terms (which are the corresponding massless distributions) cancel,

$$\Delta_m^F(y)\partial^\mu\Delta_{m_H}^F(y) - \partial^\mu\Delta_m^F(y)\Delta_{m_H}^F(y) = 0 + \mathcal{O}(m^2, m_H^2), \quad (215)$$

and the terms $\mathcal{O}(m^2, m_H^2)$ have singular order $\omega \leq -1$.

Now from (209) and (210) we obtain

$$a_0 = \frac{i}{3} C_{\text{fish}} , \quad a_1 = i \left(\frac{1}{2} \left(1 + \frac{m_H^2}{m^2} \right) - 4 \right) C_{\text{fish}} , \quad a_2 = 0 , \quad (216)$$

and from (211), (212) and (213) we get

$$b_1 = i \left(\frac{m_H^2}{m^2} - \frac{m_H^4}{m^4} \right) C_{\text{fish}} , \quad b_2 = 3i C_{\text{fish}} ; \quad (217)$$

the computation of b_2 is analogous to the computation of b_0 given in Sect. 6.1.

There are 5 terms contributing to c_1 : one term is obtained from (212) by $\varphi \leftrightarrow B$ and four terms are $\sim \omega_0 \left(T_2(\phi(x_1) \otimes \phi(x_2)) \right) \varphi(x_1) \varphi(x_2)$ where $\phi = \varphi^2, B^2, A^2$ and $\phi = \bar{u}u$. We find

$$c_1 = -i \left(6 \frac{m^2}{m_H^2} + 5 \frac{m_H^2}{m^2} \right) C_{\text{fish}} . \quad (218)$$

Coefficients of some trilinear fields: the contributions to l_0, l_3 and l_4 come from fish diagrams (without derivatives) belonging to $T_2(L_1^0 \otimes L_2)$ and from triangle diagrams (with two derivatives) belonging to $T_3(L_1^{1 \otimes 2} \otimes L_1^0)$, where $L_1^0 := m A^2 \varphi - \frac{m_H^2}{2m} (\varphi^3 + B^2 \varphi)$ and $L_1^1 := BA\partial\varphi - \varphi A\partial B$. To compute l_0 we have to take into account the terms

$$4m \omega_0 \left(T_2(A_\lambda \varphi(x_1) \otimes A_\nu \varphi(x_2)) \right) A^\lambda(x_1) A^\nu \varphi(x_2) + 1 \text{ permutation} , \quad (219)$$

$$\frac{-m_H^2}{4m} \omega_0 \left(3 T_2(\varphi^2(x_1) \otimes \varphi^2(x_2)) + T_2(B^2(x_1) \otimes B^2(x_2)) \right) \varphi(x_1) A^2(x_2) + 1 \text{ permutation} , \quad (220)$$

$$\begin{aligned} & \frac{-m_H^2}{2m} \omega_0 \left(T_3 \left((B\partial^\mu \varphi - \varphi\partial^\mu B)(x_1) \otimes (B\partial^\nu \varphi - \varphi\partial^\nu B)(x_2) \otimes (3\varphi^2 + B^2)(x_3) \right) \right) \\ & \quad \cdot A_\mu(x_1) A_\nu(x_2) \varphi(x_3) + 2 \text{ permutations} , \end{aligned} \quad (221)$$

$$\begin{aligned} & -2m \omega_0 \left(T_3(A^\mu \varphi(x_1) \otimes A\partial B(x_2) \otimes (B\partial^\lambda \varphi - \varphi\partial^\lambda B)(x_3)) \right) \\ & \quad \cdot A_\mu(x_1) \varphi(x_2) A_\lambda(x_3) + 5 \text{ permutations} , \end{aligned} \quad (222)$$

where permutations of the vertices are meant. These terms yield

$$l_0 = -4i C_{\text{fish}} + \frac{m_H^2}{m^2} (-2i C_{\text{fish}}) + \frac{m_H^2}{m^2} 2(3+1) C_{1\Delta} + 4 C_{1\Delta} = -3i C_{\text{fish}} , \quad (223)$$

where in the first step only $C_{2\Delta} = -C_{1\Delta}$ is used, and the k -th summand comes from the k -th term in (219)-(222) ($k = 1, 2, 3, 4$). In (228),(233), (240), (245) and (250) the summands are ordered correspondingly.

Turning to l_3 , the terms

$$\frac{m}{2} \omega_0 \left(T_2(A^2(x_1) \otimes A^2(x_2)) \right) \varphi(x_1) \varphi^2(x_2) + 1 \text{ permutation} , \quad (224)$$

$$\frac{m_H^4}{8m^3} \omega_0 \left(9 T_2(\varphi^2(x_1) \otimes \varphi^2(x_2)) + T_2(B^2(x_1) \otimes B^2(x_2)) \right) \varphi(x_1) \varphi^2(x_2) + 1 \text{ permutation} , \quad (225)$$

$$\frac{-m_H^2}{2m} \omega_0 \left(T_3(A\partial B(x_1) \otimes A\partial B(x_2) \otimes B^2(x_3)) \right) \varphi(x_1)\varphi(x_2)\varphi(x_3) + 2 \text{ permutations} , \quad (226)$$

$$m \omega_0 \left(T_3(A\partial B(x_1) \otimes A\partial B(x_2) \otimes A^2(x_3)) \right) \varphi(x_1)\varphi(x_2)\varphi(x_3) + 2 \text{ permutations} , \quad (227)$$

give

$$\begin{aligned} l_3 &= \frac{m^2}{m_H^2} (-8i C_{\text{fish}}) + \frac{m_H^2}{m^2} (-5i C_{\text{fish}}) + 4 C_{1\Delta} + \frac{m^2}{m_H^2} (-8 C_{2\Delta}) \\ &= i C_{\text{fish}} \left(1 - 6 \frac{m^2}{m_H^2} - 5 \frac{m_H^2}{m^2} \right) . \end{aligned} \quad (228)$$

The contributions to l_4 come from the terms

$$\frac{m}{2} \omega_0 \left(T_2(A^2(x_1) \otimes A^2(x_2)) \right) \varphi(x_1)B^2(x_2) + 1 \text{ permutation} , \quad (229)$$

$$\begin{aligned} &\frac{m_H^4}{8m^3} \omega_0 \left(3 T_2(\varphi^2(x_1) \otimes \varphi^2(x_2)) + 3 T_2(B^2(x_1) \otimes B^2(x_2)) \right) \varphi(x_1)B^2(x_2) + 1 \text{ permutation} \\ &+ \frac{m_H^4}{8m^3} 8 \omega_0 \left(T_2(\varphi B(x_1) \otimes \varphi B(x_2)) \right) B(x_1)\varphi B(x_2) + 1 \text{ permutation} , \end{aligned} \quad (230)$$

$$\begin{aligned} &\frac{-m_H^2}{2m} 3 \omega_0 \left(T_3(A\partial\varphi(x_1) \otimes A\partial\varphi(x_2) \otimes \varphi^2(x_3)) \right) B(x_1)B(x_2)\varphi(x_3) + 2 \text{ permutations} \\ &+ \frac{m_H^2}{2m} 2 \omega_0 \left(T_3(A\partial B(x_1) \otimes A\partial\varphi(x_2) \otimes \varphi B(x_3)) \right) \varphi(x_1)B(x_2)B(x_3) + 5 \text{ permutations} , \end{aligned} \quad (231)$$

$$m \omega_0 \left(T_3(A\partial\varphi(x_1) \otimes A\partial\varphi(x_2) \otimes A^2(x_3)) \right) B(x_1)B(x_2)\varphi(x_3) + 2 \text{ permutations} , \quad (232)$$

which yield

$$l_4 = \frac{m^2}{m_H^2} (-8i C_{\text{fish}}) + \frac{m_H^2}{m^2} (-(3+2)i C_{\text{fish}}) + (12-8) C_{1\Delta} + \frac{m^2}{m_H^2} (-8 C_{2\Delta}) = l_3 . \quad (233)$$

Coefficients of some quadrilinear fields: the contributions to l_5 , l_7 and l_8 come from fish diagrams (without derivatives) belonging to $T_2(L_2 \otimes L_2)$, from triangle diagrams (with two derivatives) belonging to $T_3(L_1^{1 \otimes 2} \otimes L_2)$ and from square diagrams (with four

Finally, to compute l_8 we have to take account of

$$\begin{aligned} & \varphi^2(x_1)B^2(x_2) \left[\frac{1}{4} \omega_0 \left(T_2(A^2(x_1) \otimes A^2(x_2)) \right) \right. \\ & \quad \left. + \frac{m_H^4}{8 \cdot 4 m^4} 6 \omega_0 \left(T_2(\varphi^2(x_1) \otimes \varphi^2(x_2)) + T_2(B^2(x_1) \otimes B^2(x_2)) \right) \right] + 1 \text{ permutation} \\ & + \varphi B(x_1) \varphi B(x_2) 16 \left(\frac{m_H^2}{4m^2} \right)^2 \omega_0 \left(T_2(\varphi B(x_1) \otimes \varphi B(x_2)) \right), \end{aligned} \quad (246)$$

$$\begin{aligned} & \frac{1}{2} \left[\omega_0 \left(T_3(A\partial B(x_1) \otimes A\partial B(x_2) \otimes A^2(x_3)) \right) \varphi(x_1) \varphi(x_2) B^2(x_3) \right. \\ & \quad \left. + \omega_0 \left(T_3(A\partial \varphi(x_1) \otimes A\partial \varphi(x_2) \otimes A^2(x_3)) \right) B(x_1) B(x_2) \varphi^2(x_3) \right] + 2 \text{ permutations}, \end{aligned} \quad (247)$$

$$\begin{aligned} & \frac{-m_H^2}{8m^2} 6 \left[\omega_0 \left(T_3(A\partial B(x_1) \otimes A\partial B(x_2) \otimes B^2(x_3)) \right) \varphi(x_1) \varphi(x_2) B^2(x_3) \right. \\ & \quad \left. + \omega_0 \left(T_3(A\partial \varphi(x_1) \otimes A\partial \varphi(x_2) \otimes \varphi^2(x_3)) \right) B(x_1) B(x_2) \varphi^2(x_3) \right] + 2 \text{ permutations} \\ & + \frac{m_H^2}{4m^2} 4 \omega_0 \left(T_3(A\partial B(x_1) \otimes A\partial \varphi(x_2) \otimes B\varphi(x_3)) \right) \varphi(x_1) B(x_2) \varphi B(x_3) + 5 \text{ permutations}, \end{aligned} \quad (248)$$

$$\begin{aligned} & \omega_0 \left(T_4^c(A\partial B(x_1) \otimes A\partial B(x_2) \otimes A\partial \varphi(x_3) \otimes A\partial \varphi(x_4)) \right) \\ & \quad \cdot \varphi(x_1) \varphi(x_2) B(x_3) B(x_4) + 5 \text{ permutations}, \end{aligned} \quad (249)$$

and we get

$$l_8 = -i C_{\text{fish}} \left(8 \frac{m^2}{m_H^2} + (3+2) \frac{m_H^2}{m^2} \right) - \frac{m^2}{m_H^2} (8+8) C_{2\Delta} + (12+12-16) C_{1\Delta} + \frac{m^2}{m_H^2} 8i C_{2\Box} = l_7. \quad (250)$$

Note that (227), (232), (242) and (247) can be viewed also as fish diagram contributions, since their unrenormalized versions are $\sim -g_{\mu\nu} \partial^\mu \partial^\nu D^F(x_1 - x_2) D^F(x_1 - x_3) D^F(x_2 - x_3) = i \delta(x_1 - x_2) t_{\text{fish}}^\circ(x_1 - x_3)$; however in Sect. 6.5 we treat them as triangle diagram contributions.

C Working out PGI-tree for the running interaction

In this appendix we work out PGI-tree for the interaction $\mathcal{L}(g) = z_\rho(L)(g)$, as defined after (169). We use that $\mathcal{L}(g)$ is of the form (66) with the explicit expressions (73) and (74), with unknown coefficients $e_\rho^{(j)}$ in the $L_k^{(j)}$ ($k = 0, 1, 2$) for $j \geq 1$. About the Q -vertex $\mathcal{P}^\nu(g; f)$ (67) we only know that it is of the form (75), the field polynomials $P_k^{(j)}$ are completely unknown.

It is well-known (see e.g. [DS99, Sch01, DGBSV10]) that in the inductive Epstein-Glaser construction of the time ordered products, PGI can be violated only by local

terms. Hence, we need to study only the local contributions. However, in principle the splitting of a distribution into a local and a non-local part is non-unique; hence, some caution is called for. Let x_1, \dots, x_n be the vertices of the considered connected tree diagram. Everywhere in our calculations we replace $(\partial)\square\Delta_m^F$ by $(-m^2(\partial)\Delta_m^F - i(\partial)\delta)$. Then, outside the total diagonal $x_1 = x_2 = \dots = x_n$ only terms with at least one propagator Δ_m^F , $\partial_\mu\Delta_m^F$, $\partial_\nu\partial_\mu\Delta_m^F$ and $\partial_\nu\partial_\mu\partial_\lambda\Delta_m^F$ (with no contraction of Lorentz indices) contribute. Since these terms cancel outside the total diagonal, they cancel also on the total diagonal. The remaining terms are the local terms, they are linear combinations of $\partial^a\delta(x_1 - x_n, \dots, x_{n-1} - x_n)$. We write T_{tree} for the contribution of the connected tree diagrams and $T(\dots)|_{\text{loc}}$ means the selection of the local terms. The latter is a rather delicate issue. Considering

$$\partial_\nu^x T_{\text{tree}}(P^\nu(x) \otimes L(y))|_{\text{loc}} , \quad (251)$$

there appear the following possibilities how the divergence ∂_ν^x generates local terms (cf. [DS99, DGBSV10]):

- Type 1 If $P^\nu = \partial^\nu\phi F + \dots$ and $L = \phi E + \dots$, then the contraction of $\partial^\nu\phi(x)$ with $\phi(y)$ gives a propagator $\hbar\partial^\nu\Delta_m^F(x-y)$, and on computing its divergence we find the local contribution $-i\hbar\delta(x-y)F(x)E(x)$.
- Type 2 If P^ν is as before and $L = \partial^\mu\phi E + \dots$, then analogously to type 1 we obtain the local contribution $i\hbar\partial^\mu\delta(x-y)F(x)E(y)$.
- Type 3 If $P^\nu = A^\nu F + \dots$ and $L = (\partial_\mu A^\mu)E + \dots$, then the contraction of $A^\nu(x)$ with $\partial_\mu A^\mu(y)$ gives a propagator $\hbar g^{\nu\mu}\partial_\mu\Delta_m^F(x-y)$, and we get the local contribution $-i\hbar\delta(x-y)F(x)E(x)$.

Remark C.1.

- (1) Usually interactions for spin-1 fields do not contain a $\partial_\mu A^\mu$ -field; hence, the type 3 mechanism is non-standard, however it has been used already in the application of PGI to spin-2 gauge theories [Sch01].
- (2) In the literature about PGI mostly a different normalization of the time ordered products is used (denoted by T^N in the following). Considering $S(i\mathcal{L}(g))$, where $\mathcal{L}(g)$ is of the form (66), the arguments of T^N are only the vertices $\mathcal{L}_{(1)}(x_j)$ which are of first order in g . A higher order vertex $\int dx (g(x))^n \mathcal{L}_{(n)}(x)$ ($n \geq 2$) is taken into account as a local contribution

$$n!(-i)^{n-1}\delta(x_1 - x_n, \dots, x_{n-1} - x_n)\mathcal{L}_{(n)}(x_n) \quad \text{to} \quad T_{n,\text{tree}}^N(\otimes_{j=1}^n \mathcal{L}_{(1)}(x_j)) . \quad (252)$$

Analogously a higher order Q -vertex $\int dx (g(x))^{(n-1)} \mathcal{P}_{(n)}^\nu(x) f(x)$ ($n \geq 2$) appears as a local contribution

$$(n-1)!(-i)^{n-1}\delta(x_1 - x_n, \dots, x_{n-1} - x_n)\mathcal{P}_{(n)}(x_n) \quad \text{to} \quad T_{n,\text{tree}}^N(\mathcal{P}_{(1)}(x_1) \otimes (\otimes_{j=2}^n \mathcal{L}_{(1)}(x_j))) \quad (253)$$

integrated out with $f(x_1) \prod_{j=2}^n g(x_j)$. The relation between the time ordered products T^N and T can generally be described in terms of the Main Theorem, see [Düt05, formula (2.29)].

Now we are going to work out PGI-tree. Selecting the local terms which are of order \hbar^0 and of a certain order in τ and κ , we obtain the following equations:

$$\hbar^0 \tau^0 \kappa^1 : \quad \frac{i}{\hbar} [Q, L_1(g)] \approx -(\partial P_1)(g) , \quad (254)$$

$$\begin{aligned} \hbar^0 \tau^0 \kappa^2 : \quad & \frac{i}{\hbar} [Q, L_2(g^2)] \approx -\frac{i}{2} (\partial P_2)(g^2) \\ & - \frac{i}{\hbar} \int dx dy g(x) g(y) \partial_x T_{\text{tree}}(P_1(x) \otimes L_1(y))|_{\text{loc}} , \end{aligned} \quad (255)$$

$$\begin{aligned} \hbar^0 \tau^0 \kappa^3 : \quad & 0 \approx -\frac{i}{\hbar} \int dx dy g(x) (g(y))^2 \left(\partial_x T_{\text{tree}}(P_1(x) \otimes L_2(y))|_{\text{loc}} \right. \\ & \left. + \frac{1}{2} \partial_y T_{\text{tree}}(L_1(x) \otimes P_2(y))|_{\text{loc}} \right) , \end{aligned} \quad (256)$$

$$\hbar^0 \tau^1 \kappa^2 : \quad \frac{i}{\hbar} [Q, L_0^{(1)}(g^2)] \approx -\frac{1}{2} (\partial P_0^{(1)})(g^2) , \quad (257)$$

$$\begin{aligned} \hbar^0 \tau^1 \kappa^3 : \quad & \frac{i}{\hbar} [Q, L_1^{(1)}(g^3)] \approx -\frac{1}{3} (\partial P_1^{(1)})(g^3) \\ & - \frac{i}{\hbar} \int dx dy (g(x))^2 g(y) \left(\partial_y T_{\text{tree}}(L_0^{(1)}(x) \otimes P_1(y))|_{\text{loc}} \right. \\ & \left. + \frac{1}{2} \partial_x T_{\text{tree}}(P_0^{(1)}(x) \otimes L_1(y))|_{\text{loc}} \right) , \end{aligned} \quad (258)$$

$$\begin{aligned} \hbar^0 \tau^1 \kappa^4 : \quad & \frac{i}{\hbar} [Q, L_2^{(1)}(g^4)] \approx -\frac{1}{4} (\partial P_2^{(1)})(g^4) \\ & - \frac{i}{\hbar} \int dx dy (g(x))^3 g(y) \left(\frac{1}{3} \partial_x T_{\text{tree}}(P_1^{(1)}(x) \otimes L_1(y))|_{\text{loc}} \right. \\ & \quad \left. + \partial_y T_{\text{tree}}(L_1^{(1)}(x) \otimes P_1(y))|_{\text{loc}} \right) \\ & - \frac{i}{2\hbar} \int dx dy (g(x))^2 (g(y))^2 \left(\partial_x T_{\text{tree}}(P_0^{(1)}(x) \otimes L_2(y))|_{\text{loc}} \right. \\ & \quad \left. + \partial_y T_{\text{tree}}(L_0^{(1)}(x) \otimes P_2(y))|_{\text{loc}} \right) \\ & + \frac{1}{\hbar^2} \int dy dx_1 dx_2 g(y) (g(x_1))^2 g(x_2) \partial_y T_{\text{tree}}(P_1(y) \otimes L_0^{(1)}(x_1) \otimes L_1(x_2))|_{\text{loc}} , \end{aligned} \quad (259)$$

$$\begin{aligned}
\hbar^0 \tau^1 \kappa^5 : \quad 0 \approx & -\frac{i}{\hbar} \int dx dy (g(x))^3 (g(y))^2 \left(\frac{1}{3} \partial_x T_{\text{tree}}(P_1^{(1)}(x) \otimes L_2(y)) \Big|_{\text{loc}} \right. \\
& + \frac{1}{2} \partial_y T_{\text{tree}}(L_1^{(1)}(x) \otimes P_2(y)) \Big|_{\text{loc}} \Big) \\
& - \frac{i}{\hbar} \int dx dy g(x) (g(y))^4 \left(\partial_x T_{\text{tree}}(P_1(x) \otimes L_2^{(1)}(y)) \Big|_{\text{loc}} \right. \\
& + \frac{1}{4} \partial_y T_{\text{tree}}(L_1(x) \otimes P_2^{(1)}(y)) \Big|_{\text{loc}} \Big) \\
& + \frac{1}{\hbar^2} \int dy dx_1 dx_2 g(y) (g(x_1))^2 (g(x_2))^2 \partial_y T_{\text{tree}}(P_1(y) \otimes L_0^{(1)}(x_1) \otimes L_2(x_2)) \Big|_{\text{loc}} .
\end{aligned} \tag{260}$$

This list contains *all* local terms of (169) which are of order $\hbar^0 \tau^0 \kappa^l$ or $\hbar^0 \tau^1 \kappa^l$ for l arbitrary. On computing the terms appearing in this list, we replace $\square\phi$ by $-m_\phi^2 \phi$.

- *PGI-equations* (254)-(256). The τ^0 -equations express PGI-tree for the ($\rho = 1$)-theory, they have a unique solution for P_1 and P_2 given in (70) (cf. [ADS97, GB11]).
- *PGI-equation* (257). (*Tree diagrams with 2 external lines.*) Throughout this appendix we use the notation $e \log \rho := e_\rho^{(1)}$ (206). With that (257) is equivalent to

$$a_1 - a_2 + b_2 - c_2 = 0 \quad \wedge \quad b_2 + b_0 - b_1 + c_2 = 0 \tag{261}$$

and a non-unique expression for $P_0^{(1)}$:

$$\frac{1}{2} P_0^{(1)\nu} = (c_2 + a_2) m^2 A^\nu u + (b_2 + b_0) (\sigma m u \partial^\nu B + (1 - \sigma) m B \partial^\nu u) , \tag{262}$$

where $\sigma \in \mathbb{C}$ is an arbitrary number.

- *PGI-equation* (258). (*Tree diagrams with 3 external lines.*) A type 3 term appears only in $\partial_y T_{\text{tree}}(L_0^{(1)}(x) \otimes P_1(y)) \Big|_{\text{loc}}$. The equation (258) is equivalent to the following relations: $P_1^{(1)}$ is of the form

$$\frac{1}{3} P_1^{(1)\nu} = \alpha \varphi B \partial^\nu u + \beta \varphi u \partial^\nu B + \gamma u B \partial^\nu \varphi + \lambda m A^\nu u \varphi , \quad \alpha, \beta, \gamma, \lambda \in \mathbb{C} ; \tag{263}$$

and

$$m A \partial u \varphi : \quad 0 = -2l_0 + l_2 + \lambda - \frac{\sigma}{3} (b_2 + b_0) - \frac{2}{3} b_2 , \tag{264}$$

$$m A u \partial \varphi : \quad 0 = -l_1 + \lambda + \frac{5\sigma}{3} (b_2 + b_0) - \frac{2}{3} b_2 , \tag{265}$$

$$m \partial A u \varphi : \quad 0 = \lambda + \frac{2\sigma}{3} (b_2 + b_0) - a_2 + \frac{1}{3} b_2 , \tag{266}$$

$$B \partial u \partial \varphi : \quad 0 = -l_1 + \alpha + \gamma + \frac{2}{3} c_0 , \tag{267}$$

$$\partial B \partial u \varphi : \quad 0 = l_2 + \alpha + \beta - \frac{2}{3} b_0 , \tag{268}$$

$$\partial B u \partial \varphi : \quad 0 = \beta + \gamma + \frac{2}{3} c_0 - \frac{2}{3} b_0 , \tag{269}$$

$$\begin{aligned}
u B \varphi : \quad 0 = & m_H^2 (l_4 - \gamma - \sigma (b_2 + b_0) - c_1 + \frac{1}{3} c_0) \\
& + m^2 (-\alpha - \beta - (1 - \sigma)(b_2 + b_0) + b_1 - \frac{1}{3} b_0) .
\end{aligned} \tag{270}$$

The equations (264)-(270) are obtained by setting the coefficient of the indicated field monomial to zero.

- *PGI-equation (259). (Tree diagrams with 4 external lines.)* A type 3 contribution appears only in $\partial_y T_{\text{tree}}(L_0^{(1)}(x) \otimes P_4(y))|_{\text{loc}}$. There is only one type 1 contribution coming from a contraction of ∂u with \tilde{u} , namely in $\partial_x T_{\text{tree}}(P_1^{(1)}(x) \otimes L_1(y))|_{\text{loc}}$. The last term in (259) is the most difficult one; we explain the computation: the local contributions come from terms of the form

$$\frac{1}{\hbar^2} \int dy dx_1 dx_2 g(y) (g(x_1))^2 g(x_2) \partial_\nu^y T_{\text{tree}}((G \partial^\nu \phi)(y) \otimes \frac{1}{2} (\partial \phi)^2(x_1) \otimes (F_\tau \partial^\tau \phi)(x_2))|_{\text{loc}}, \quad (271)$$

where $\phi = B$ or $\phi = \varphi$. The contraction of $\partial^\nu \phi(y)$ with $\partial^\mu \phi(x_1)$ is of type 2; the contraction of $\partial_\mu \phi(x_1)$ with $\partial^\tau \phi(x_2)$ gives a propagator $i\hbar \partial_\mu \partial^\tau \Delta^F(x_1 - x_2)$. With that (271) is equal to

$$\begin{aligned} &= -i \int dy dx_1 dx_2 g(y) (g(x_1))^2 g(x_2) G(y) \partial^\mu \delta(y - x_1) \partial_\mu \partial^\tau \Delta^F(x_1 - x_2) F_\tau(x_2)|_{\text{loc}} \\ &= i \int dy dx_2 \frac{\partial_y^\mu (g(y))^3}{3} g(x_2) G(y) \partial_\mu \partial^\tau \Delta^F(y - x_2) F_\tau(x_2)|_{\text{loc}} \\ &= \frac{-1}{3} \int dy dx_2 (g(y))^3 g(x_2) G(y) \partial^\tau \delta(y - x_2) F_\tau(x_2) \\ &= \frac{1}{3} \int dy \left(\frac{3}{4} \partial_\tau (g(y))^4 G(y) F_\tau(y) + (g(y))^4 \partial^\tau G(y) F_\tau(y) \right) \\ &= \int dy (g(y))^4 \left(\frac{1}{12} \partial^\tau G(y) F_\tau(y) - \frac{1}{4} G(y) \partial^\tau F_\tau(y) \right), \end{aligned} \quad (272)$$

where non-local terms are omitted. If the x_2 -vertex is of the simpler form $(F\phi)(x_2)$, then $\partial^\tau \Delta^F(x_1 - x_2) F_\tau(x_2)$ is replaced by $-\Delta^F(x_1 - x_2) F(x_2)$ and it results $\frac{1}{3} \int dy (g(y))^4 G(y) F(y)$.

Proceeding as for (258), the PGI-equation (259) is equivalent to the following: $P_2^{(1)}$ is of the form

$$\frac{1}{4} P_2^{(1)\nu} = \Upsilon u \varphi^2 A^\nu + \Xi u B^2 A^\nu, \quad \Upsilon, \Xi \in \mathbb{C}, \quad (273)$$

and

$$A^2 A \partial u : 0 = l_{11} , \quad (274)$$

$$B \varphi^2 u : 0 = -m\alpha + \frac{m_H^2}{2m}(l_8 - 3l_3 + 2l_4 - \sigma(b_2 + b_0) - 3\gamma - 2\beta + c_0 - \frac{2}{3}b_0) , \quad (275)$$

$$m A^2 B u : 0 = -l_6 + l_0 + \sigma(b_2 + b_0) + \gamma - \frac{1}{3}c_0 , \quad (276)$$

$$\frac{m_H^2}{2m} B^3 u : 0 = l_9 - l_4 - \sigma(b_2 + b_0) - \gamma + \frac{1}{3}c_0 , \quad (277)$$

$$A \varphi^2 \partial u : 0 = \Upsilon - l_5 + \frac{3}{4}l_2 - \frac{1}{4}\beta - \frac{1}{12}b_0 , \quad (278)$$

$$\partial A \varphi^2 u : 0 = \Upsilon - \frac{1}{4}l_2 - \frac{1}{2}a_2 + \frac{3}{4}\beta + \frac{1}{4}b_0 , \quad (279)$$

$$2A u \varphi \partial \varphi : 0 = \Upsilon + \frac{1}{4}l_2 - \frac{1}{2}l_1 + \frac{3}{4}\beta + \frac{1}{4}b_0 , \quad (280)$$

$$A B^2 \partial u : 0 = \Xi - l_6 + \frac{3}{4}l_1 + \frac{1}{4}\gamma - \frac{1}{12}c_0 , \quad (281)$$

$$\partial A B^2 u : 0 = \Xi - \frac{1}{4}l_1 - \frac{1}{2}a_2 - \frac{3}{4}\gamma + \frac{1}{4}c_0 , \quad (282)$$

$$2A u B \partial B : 0 = \Xi + \frac{1}{4}l_1 - \frac{1}{2}l_2 - \frac{3}{4}\gamma + \frac{1}{4}c_0 . \quad (283)$$

- *PGI-equation (260). (Tree diagrams with 5 external lines.)* Note that

$$\partial_y T_{\text{tree}}(L_1^{(1)}(x) \otimes P_2(y))|_{\text{loc}} = 0 = \partial_y T_{\text{tree}}(L_1(x) \otimes P_2^{(1)}(y))|_{\text{loc}} , \quad (284)$$

since $A^\nu(y)$ (appearing in $P_2(y)$ and $P_2^{(1)}(y)$) has no partner field $\partial_\mu A^\mu(x)$ which is needed for a type 3 contribution. Proceeding as above we get

$$u B A^2 \varphi : 0 = l_5 - l_6 + \beta + \gamma + \frac{1}{3}(b_0 - c_0) , \quad (285)$$

$$\frac{m_H^2}{2m^2} u \varphi^3 B : 0 = l_8 - l_7 - \beta - \gamma + \frac{1}{3}(c_0 - b_0) , \quad (286)$$

$$\frac{m_H^2}{2m^2} u \varphi B^3 : 0 = l_9 - l_8 - \beta - \gamma + \frac{1}{3}(c_0 - b_0) . \quad (287)$$

The system of equations (261), (264)-(270), (274)-(283) and (285)-(287) contains a lot of redundancies; the most general solution is given in (171). To complete this result we add

$$\begin{aligned} \alpha &= 0 , & \beta &= -l_1 + \frac{2b_0}{3} , & \gamma &= l_1 - \frac{2c_0}{3} , & \lambda &= b_0 + c_2 - b_1 - \frac{2l_1}{3} , \\ \Upsilon &= l_1 - \frac{3b_0}{4} , & \Xi &= l_1 - \frac{3c_0}{4} \end{aligned} \quad (288)$$

and the relation determining σ ,

$$\sigma(b_1 - c_2) = l_1 - b_0 + b_1 - c_2 . \quad (289)$$

The most general solution of the BRST-condition (54) (given in (59)-(61) and for the pertinent Q -vertex see (170)) is a *true* subset of the PGI-tree solution computed here, due to Remark 7.1. This subset property is a good check of the calculations in this appendix.

The result (171) gives the restrictions from PGI-tree on the 1-loop coefficients $e_\rho^{(1)}$. The corresponding restrictions on the higher loop coefficients $e_\rho^{(2)}$, $e_\rho^{(3)}$, ... can be obtained by continuing the calculations in this appendix: one has to select the local terms of (169) which are of order $\hbar^0 \tau^r \kappa^l$ for l arbitrary and $r = 2, 3, \dots$.

Remark C.2. We now are able to see, why the claim (134) holds true. First note that PC for $S(iz_\rho(L)(g))$ (133) implies PC for the *connected* time-ordered products:

$$\lim_{\varepsilon \downarrow 0} [Q, S^c(iz_\rho(L)(g_\varepsilon))]_\star \approx 0, \quad (290)$$

this follows analogously to (169). Now, using the τ -trick in this equation, the terms $\sim \tau^0$ vanish separately, because they are the $U(1)$ -Higgs model, which fulfills PGI and, hence, also (290). Therefore, taking $\tau = \hbar$ into account, there cannot be a cancellation of terms $\sim \hbar^0 \tau^1 \kappa^4$ with terms $\sim \hbar^1 \tau^0 \kappa^4$; hence, the terms $\sim \hbar^0 \tau^1 \kappa^4$ (which are tree-terms) must fulfill (290) separately. Moreover, as explained above, the non-local connected tree terms fulfill PGI separately and, hence, they fulfill also (290) separately. Now, as we see from (274), there is only one local connected (tree) term $\sim \hbar^0 \tau^1 \kappa^4 A^2 A \partial u$ contributing to the l.h.s. of (290), namely the r.h.s. of (134); therefore, the latter must be ≈ 0 individually.

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