RENORMALIZATION OF DETERMINANTS LINES IN QUANTUM FIELD THEORY.

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Abstract. On a compact manifold M, we consider the affine space $\mathcal{A}$ of non self–adjoint perturbations of some invertible elliptic operator acting on sections of some Hermitian bundle, by some differential operator of lower order.

We construct and classify all complex analytic functions on the Fréchet space $\mathcal{A}$ vanishing exactly over non invertible elements, having minimal order and which are obtained by local renormalization, a concept coming from quantum field theory, called *renormalized determinants*. The additive group of local polynomial functionals of finite degrees acts freely and transitively on the space of renormalized determinants. We provide different representations of the renormalized determinants in terms of spectral zeta determinants, Gaussian Free Fields, infinite product and renormalized Feynman amplitudes in perturbation theory in position space à la Epstein–Glaser.

Specializing to the case of Dirac operators coupled to vector potentials and reformulating our results in terms of determinant line bundles, we prove our renormalized determinants define some complex analytic trivializations of some holomorphic line bundle over $\mathcal{A}$ relating our results to a conjectural picture from some unpublished notes by Quillen [56] from April 1989.

1. Introduction.

Let (M,g) be a smooth, closed, compact Riemannian manifold. The aim of the present paper is to study the analytical properties and the renormalization of a class of functional determinants defined on some affine space $\mathcal{A}$ of non self adjoint perturbations $\Delta+V$ of some given invertible self adjoint generalized Laplacian Δ acting on some fixed Hermitian bundle $E\mapsto M$ where $V\in C^\infty(End(E))$ is a smooth potential, or perturbations of the form $D+A$ of some invertible twisted Dirac operator $D: C^\infty(E_+)\mapsto C^\infty(E_-)$ acting between Hermitian bundles $E_\pm$ by some term $A\in C^\infty(Hom(E_+,E_-))$. We consider lower order perturbations since A and V are local operators of order 0.

1.0.1. *Quantum field theory interacting with some external potential.* Let us briefly give the physical motivations underlying our results which are stated in purely mathematical terms. The reader uninterested by the physics can safely skip this part. Inspired by recent works in mathematical physics [18, 19, 20, 21, 30, 29] and classical works of Schwinger [62] [42, Chapter 4 p. 163], our original purpose is to understand the problem of renormalization of some Euclidean quantum field ϕ defined on M interacting with a classical external field which is not quantized [1]. For instance, consider the Laplace–Beltrami operator $\Delta: C^\infty(M)\mapsto C^\infty(M)$

[1] sometimes called *background field* in the physical litterature

defined from the metric g on M, corresponding to the Dirichlet action functional :

$$S(\phi)=\int_M (\phi\Delta\phi)\, dv \tag{1.1}$$

and their perturbations by some external potential $V\in C^\infty(M,\mathbb{R}_{\geqslant 0})$ which corresponds to the perturbed Dirichlet action functional

$$S(\phi)=\int_M \left(\phi\Delta\phi+V\phi^2\right) dv. \tag{1.2}$$

A classical problem in quantum field theory is to define the *partition function* of a theory, usually represented by some ill–defined functional integral,

$$Z\left(V\right)=\int [d\phi]\exp\left(-\frac{1}{2}\int_M \left(\phi\Delta\phi+V\phi^2\right) dv\right) \tag{1.3}$$

in the bosonic case where $V\in C^\infty(M,\mathbb{R}_{>0})$ plays the role of a position dependent mass which is viewed as an external field coupled to the Gaussian Free Field ϕ to be quantized. The external field can also be the metric g [1] in the study of *gravitational anomalies* in the physics litterature or *gauge fields*, which is the physical terminology for connection 1–forms, in the study of *chiral anomalies*.

In fact, according to Stora [70, 52], the physics of chiral anomalies [26, 52, 1] can be understood in the case where we have a fermion field which is quantized interacting with some *gauge field* which is treated as an external field. Consider the quadratic Lagrangian $\Psi_-D_A\Psi_+$ where $\Psi_\pm$ are *chiral* fermions, D_A is a twisted half–Dirac operator acting from sections of positive spinors S_+, to negative spinors S_-, see example 1 below for a precise definition of D_A. In this case, the corresponding ill–defined functional integral reads

$$Z\left(A\right)=\int [\mathcal{D}\Psi_-\mathcal{D}\Psi_+]\exp\left(-\int_M \langle\Psi_-,D_A\Psi_+\rangle\right)$$

where we are interested on the dependence in the gauge field A.

1.0.2. *Functional determinants in geometric analysis.* The above two problems can be formulated as the mathematical problem of constructing functional determinants $V\mapsto \det\left(\Delta+V\right)$ and $A\mapsto \det\left(D^*\left(D+A\right)\right)$ with nice functional properties where we are interested in the dependence in the external potentials V and A. In global analysis, functional determinants also appear in the study of the analytic torsion by Ray–Singer [58] and more generally as metrics of determinant line bundles as initiated by Quillen [57, 3] where he considered some affine space $\mathcal{A}$ of Cauchy-Riemann operators $D+\omega$ acting on some fixed vector bundle $E\mapsto X$ over a compact Riemann surface X where D is fixed and the perturbation ω lives in the linear space of $(0,1)$-forms on X with values in the bundle $End(E)$. The metrics on X and in E induce a metric in the determinant (holomorphic line) bundle $\det\left(\mathrm{Ind}(D)\right)=\Lambda^{top}\ker(D)^*\otimes\Lambda^{top}\mathrm{coker}(D)$ over $\mathcal{A}$. As Quillen showed in [57], if this metric in the bundle $\det(\mathrm{Ind}(D))$ is divided by the function $\det_\zeta\left(D^*D\right)$ (here $\det_\zeta$ is the zeta regularized determinant the Laplacian D^*D), then the canonical curvature form of this metric, the first Chern form, coincides with the symplectic form of the natural Kähler metric on $\mathcal{A}$. An important consequence of the above observation, stated as a [57, Corollary p. 33], is that if one multiplies the Hermitian metric on $\det(\mathrm{Ind}(D))$ by e^q where q is the natural Kähler metric on $\mathcal{A}$, then the corresponding Chern connection is **flat**. From the contractibility of $\mathcal{A}$, one deduces the existence of a **global holomorphic trivialization** of $\det(\mathrm{Ind}(D)) \mapsto \mathcal{A}$ and the image of the canonical section of $\det(\mathrm{Ind}(D))$ by this trivialization is an analytic function on $\mathcal{A}$ vanishing over non–invertible elements.

Building on some ideas from the work of Perrot [52, 54, 53] and some unpublished notes from Quillen's notebook [56] [2], we attempt to relate the problem of constructing renormalized determinants with the construction of holomorphic trivializations of determinant line bundles over some affine space $\mathcal{A}$ of perturbations of some fixed operator by some differential operator of lower order which plays the role of the external potential.

1.0.3. *Quillen's conjectural picture.* In some notes on the 30th of April 1989 [56, p. 282], with the motivation to make sense of the technique of adding local counterterms to the Lagrangian used in renormalized perturbation theory, Quillen proposed to give an interpretation of QFT partition functions in terms of determinant line bundles over the space of Dirac operator coupled to a gauge potential drawing a direct connection between the two subjects. The approach he outlined insists on constructing *complex analytic trivializations* of the determinant line bundle without mentioning any construction of Hermitian metrics on the line bundle which seems different from the original approach he pioneered [57] and the Bismut–Freed [3] definition of determinant line bundle for families of Dirac operators.

To explain this connection, we recall that for a pair $\mathcal{H}_0, \mathcal{H}_1$ of complex Hilbert spaces, there is a **canonical holomorphic line bundle Det** $\longmapsto \mathrm{Fred}_0(\mathcal{H}_0, \mathcal{H}_1)$ where $\mathrm{Fred}_0(\mathcal{H}_0, \mathcal{H}_1)$ is the space of Fredholm operators of **index** 0 with fiber $\mathbf{Det}_B \simeq \Lambda^{top}\ker(B)^* \otimes \Lambda^{top}\mathrm{coker}(B)$ and canonical section σ [57, p. 32] [65, p. 137–138]. Consider the **complex affine space** $\mathcal{A} = D + C^\infty(Hom(E_+, E_-))$ of perturbations of some fixed invertible Dirac operator D by some **differential operator** $A \in C^\infty(Hom(E_+, E_-))$ of order 0. We denote by $L^2(E_+)$ the space of L^2 sections of E_+. Then the map $\iota\ : D + A \in \mathcal{A} \mapsto Id + D^{-1}A \in \mathrm{Fred}_0\left(L^2(E_+), L^2(E_+)\right)$ allows to pull–back the holomorphic line bundle **Det** as a holomorphic line bundle $\mathcal{L} = \iota^*\mathbf{Det} \mapsto \mathcal{A}$ over the affine space $\mathcal{A}$ with canonical section $\underline{\det} = \iota^*\sigma$. We insist that we view $C^\infty(Hom(E_+, E_-))$ as a $\mathbb{C}$–vector space, elements in $C^\infty(Hom(E_+, E_-))$ need not preserve Hermitian structures. According to Quillen [56, p. 282], the relation with QFT goes as follows, *one gives a meaning to the functional integrals*

$$A \ \mapsto \ \int \mathcal{D}\Psi_+\mathcal{D}\Psi_- e^{-\int_M \langle \Psi_-, D_A\Psi_+\rangle}, \tag{1.4}$$

[3] *by trivializing the determinant line* . In other words, denoting by $\mathcal{O}(\mathcal{L})$ (resp $\mathcal{O}(\mathcal{A})$) the holomorphic sections (resp functions) of $\mathcal{L}$ (resp on $\mathcal{A}$), we aim at constructing a **holomorphic trivialization** of the line bundle $\tau : \mathcal{O}(\mathcal{L}) \longmapsto \mathcal{O}(\mathcal{A})$ so that the **image** $\tau(\underline{\det})$ of the canonical section $\underline{\det}$ by this trivialization is an entire function $f(P + \mathcal{V})$ on $\mathcal{A}$ vanishing exactly over the set Z of non invertible elements of $\mathcal{A}$. In some sense, this should generalize the original construction of Quillen of the holomorphic trivialization of the determinant line

[2] made available by the Clay foundation at http://www.claymath.org/publications/quillen-notebooks

[3] D_A is the Dirac operator coupled to the gauge potential A as described in [68, section 3 p. 325]

bundle over the space of Cauchy–Riemann operators [57]. Furthermore, Quillen [56, p. 284] writes :

These considerations lead to the following conjectural picture. Over the space $\mathcal{A}$ of gauge fields there should be a principal bundle for the additive group of polynomial functions of degree $\leqslant d$ where d bounds the trace which have to be regularized. The idea is that near each $A \in \mathcal{A}$ we should have a well–defined trivialization of $\mathcal{L}$ up to $\exp$ of such a polynomial. Moreover, we should have a flat connection on this bundle.

To address this conjectural picture, we follow a backward path compared to [57]. Instead of constructing some Hermitian metric then a flat connection on $\mathcal{L} \mapsto \mathcal{A}$ to trivialize the bundle, we prove in Theorem 2 an infinite dimensional analog of the classical Hadamard factorization Theorem 4 in complex analysis. We classify all determinant like functions such that :

- They are entire functions on $\mathcal{A}$ with minimal growth at infinity, a concept with is defined below as the *order* of the entire function, vanishing over non invertible elements in $\mathcal{A}$.
- Their derivatives should satisfy some simple identities reminiscent of the situation for the usual determinant in finite dimension.
- They are obtained from a renormalization by subtraction of local counterterms, a concept coming from quantum field theory which will be explained below in paragraph 2.0.8.

Trivializations of $\mathcal{L}$ are simply obtained by dividing the canonical section of $\mathcal{L}$ by the constructed determinant like functions as showed in Theorem 3. A nice consequence of our investigation is a new factorization formula for zeta regularized determinant 2.6 in terms of Gohberg–Krein's regularized determinants. We show that our renormalized determinants are not canonical and there are some ambiguities involved in their construction of the form $\exp(P)$ where P is a local polynomial functional of A. Then we show that the additive group of local polynomial functionals of A, sometimes called the renormalization group of Stüeckelberg–Petermann in the physics litterature, acts freely and transitively on the space of renormalized determinants we construct.

2. Main results.

2.0.4. *Geometric setting.* In the present paragraph, we fix once and for all the assumptions and the general geometric framework of the main Theorems (1),(2),(3) and that we shall use in the sequel. For $E \mapsto M$ some smooth Hermitian vector bundle over the compact manifold M, we denote by $C^\infty(E)$ smooth sections of E. An operator $\Delta : C^\infty(E) \mapsto C^\infty(E)$ is called generalized Laplacian if the principal part of Δ is positive definite, symmetric (i.e. formally self–adjoint) and diagonal with symbol $g_{\mu\nu}(x)\xi^\mu\xi^\nu \otimes Id_{E_x}$ in local coordinates at $(x;\xi) \in T^*M$ where g is the Riemannian metric on M. We are interested in the following two geometric situations :

Definition 2.1 (Bosonic case)**.** *Let (M,g) be a smooth, closed, compact Riemannian manifold and E some Hermitian bundle on M. We consider the complex affine space $\mathcal{A}$ of perturbations of the form $\Delta+V$ where V is a smooth endomorphism $V\in C^\infty(End(E))$, and $\Delta : C^\infty(E)\mapsto C^\infty(E)$ is an invertible generalized Laplacian. The element $V\in C^\infty(End(E))$ is treated as external potential.*

Definition 2.2 (Fermionic case)**.** *Let (M,g) be a smooth, closed, compact Riemannian manifold. Slightly generalizing the framework described in* [68, section 3 p. 325–327] *in the spirit of* [2, def 3.36 p. 116]*, we are given some pair of isomorphic Hermitian vector bundles (E_+,E_-) of finite rank over M and an invertible, elliptic first order differential operator $D : C^\infty(E_+)\mapsto C^\infty(E_-)$ such that both $DD^* : C^\infty(E_-)\mapsto C^\infty(E_-)$ and $D^*D : C^\infty(E_+)\mapsto C^\infty(E_+)$ are generalized Laplacians where D^* is the adjoint of D induced by the metric g on M and the Hermitian metrics on the bundles (E_+,E_-). We consider the complex affine space $\mathcal{A}$ of perturbations $D+A : C^\infty(E_+)\mapsto C^\infty(E_-)$ where $A\in C^\infty(Hom(E_+,E_-))$.*

Recall that in both cases, we perturb some fixed operator by a **local** operator of order 0. In the sequel, for a pair (E,F) of bundles over M, we always identify an element $\mathcal{V}\in C^\infty(Hom(E,F))$, which is a C^∞ section of the bundle $Hom(E,F)$ with the **corresponding linear operator** $\mathcal{V} : C^\infty(E)\mapsto C^\infty(F)$, in the scalar case this boils down to identifying a function $V\in C^\infty(M)$ with the multiplication operator $\varphi\in C^\infty(M)\mapsto V\varphi\in C^\infty(M)$. To avoid repetitions and to stress the similarities between bosons and fermions, we will often denote in the sequel(for problem 2.7, Theorems 2 and 3) $\mathcal{A}=P+C^\infty(Hom(E,F))$ for the affine space of perturbations of $P=\Delta$ of degree 2, $E=F$ in the bosonic case and of $P=D$ of degree 1, $E=E_+,F=E_-$ in the fermionic case.

We next give an important example from the physics litterature which fits exactly in the fermionic situation :

Example 1 (Quantized Spinor fields interacting with gauge fields)**.** *Assume (M,g) is* ***spin of even dimension*** *whose scalar curvature is nonnegative and positive at some point on M. For example $M=\mathbb{S}^{2n}$ with metric g close to the round metric. Then it is well–known that the complex spinor bundle $S\mapsto M$ splits as a direct sum $S=S_+\oplus S_-$ of isomorphic hermitian vector bundles, the classical Dirac operator $D : C^\infty(S)\mapsto C^\infty(S)$ is a formally self–adjoint, elliptic operator which is invertible by the positivity of the scalar curvature thanks to the Lichnerowicz formula* [46, Cor 8.9 p. 160].

Consider an external hermitian bundle $\mathcal{F}\mapsto M$ which is coupled to S by tensoring $(S_+\oplus S_-)\otimes\mathcal{F}=E_+\oplus E_-$. For any Hermitian connection $\nabla^{\mathcal{F}}$ on $\mathcal{F}$, we define the twisted Dirac operator $D_{\mathcal{F}} : C^\infty(S\otimes\mathcal{F})\mapsto C^\infty(S\otimes\mathcal{F})$, which is a first order differential operator of degree 1 w.r.t. the $\mathbb{Z}_2$ grading, $D_{\mathcal{F}}=c(e_i)\left(\nabla^S_{e_i}\otimes Id+Id\otimes\nabla^{\mathcal{F}}_{e_i}\right)$ near $x\in M$ where $(e_i)_i$ is a local orthonormal frame of TM near x, $c(e_i)$ is the Clifford action of the local orthonormal frame $(e_i)_i$ of TM on S. In the study of chiral anomalies, one is interested by the half–Dirac operator $D : C^\infty(S_+\otimes\mathcal{F})\mapsto C^\infty(S_-\otimes\mathcal{F})$. If (M,g) has ***positive scalar curvature*** *and the curvature of $\nabla^{\mathcal{F}}$ is small enough then* $\dim\ker(D)=0$ *and* $Ind(D)=0$ [46, prop 6.4 p. 315]*. Two connections on $\mathcal{F}$ differ by an element $\mathfrak{A}\in\Omega^1(M,End(\mathcal{F}))$. So we may define* *perturbations $D+A$ of our half–Dirac operator D, induced by perturbations of $\nabla^{\mathcal{F}}$, of the form*

$$\boxed{D+A = c(e_i)\left(\nabla^S_{e_i}\otimes Id + Id\otimes\left(\nabla^{\mathcal{F}}_{e_i}+\mathfrak{A}(e_i)\right)\right).} \tag{2.1}$$

2.0.5. *An analytic reformulation of Quillen's conjectural picture.* In our setting, we attempt to reformulate Quillen's question as a problem of constructing an entire function with prescribed zeros in infinite dimension generalizing the Fredholm determinant. A naive approach discussed in [56] would be to consider the Fredholm determinant $\det_F\left(I+D^{-1}A\right)$ where for small A, we expect that $\log\det_F\left(I+D^{-1}A\right) = \sum_{k=1}^\infty \frac{(-1)^{k+1}}{k} Tr\left((D^{-1}A)^k\right)$. However the operator $D^{-1}A$ is a pseudodifferential operator of order -1, hence for $k>d$ the power $(D^{-1}A)^k$ is trace class hence the traces $Tr\left((D^{-1}A)^k\right)$ are well–defined whereas for $k\leqslant d$ these traces are ill–defined and often **divergent** as usual in QFT. We will later see how to deal with these divergent traces in Theorem 2.

The usual method to construct functional determinants is the zeta regularization pioneered by Ray–Singer [58] in their seminal work on analytic torsion and rely on spectral or pseudodifferential methods [33, 64], see also [51, 63] for some nice recent reviews of various methods to regularize traces and determinants. Let us recall the definition of such analytic regularization (see [5, section 3 p. 203] for a very nice summary on the main results on zeta determinants) :

Definition 2.3 (Spectral zeta regularization). *Let M be a smooth, closed compact manifold and $E\mapsto M$ some Hermitian bundle. For every perturbation of the form $\Delta+V: C^\infty(E)\mapsto C^\infty(E)$ of an invertible symmetric generalized Laplacian Δ by some differential operator V of order 1, we denote by $\sigma(\Delta+V)\subset\mathbb{C}$ its spectrum which is known to be discrete by ellipticity. If $\sigma(\Delta+V)\cap\mathbb{R}_{\leqslant 0}=\emptyset$ then using the classical determination of the logarithm on $\mathbb{C}\setminus\mathbb{R}_{\leqslant 0}$, the spectral zeta function is defined as $\zeta_{\Delta+V}(s)=\sum_{\lambda\in\sigma(\Delta+V)}\lambda^{-s} = Tr\left((\Delta+V)^{-s}\right)$ [4] which has meromorphic continuation to the whole complex plane by the work of Seeley [33, 64] and is holomorphic near $s=0$. The zeta determinant $\det_\zeta$ is defined as : $\det_\zeta(\Delta+V) = \exp\left(-\zeta'_{\Delta+V}(0)\right)$.*

Let us comment the above definition. For the moment, the definition of $\det_\zeta$ restricts to operators whose spectrum does not meet the negative reals $\mathbb{R}_{\leqslant 0}$ which forms an open subset in $\mathcal{A}$[5]. Then our factorization formula 2.6 will imply that the zeta determinant defined above extends to all operators in $\mathcal{A}$, even to some non smooth perturbations of order 0 of a given Δ.

We next give the particular definitions of the zeta functions in the bosonic and fermionic cases :

Definition 2.4 (Zeta determinants for bosons and fermions.). *We use the geometric setting for bosons and fermions defined in paragraph 2.0.4. For bosons, we define the corresponding*

[4] The equality with the trace follows from Lidskii's Theorem

[5] In fact, the leading symbol of Δ is nonnegative, self–adjoint hence its spectrum except finite number of eigenvalues is contained in angular sectors of the form $\{\rho e^{i\theta}, 0\leqslant\rho\leqslant+\infty, \theta\in[-\varepsilon,\varepsilon]\}$ [5, p. 206][64] but we impose a stronger condition for the moment

zeta determinant as a map

$$V \in C^\infty(End(E)) \mapsto det_\zeta(\Delta + V). \tag{2.2}$$

For fermions, following [68, p. 329], *we define the corresponding zeta determinant as a map*

$$A \in C^\infty(Hom(E_+, E_-)) \longmapsto det_\zeta\left(D^*(D+A)\right). \tag{2.3}$$

Finally before we state our first result, we define formal products of Schwartz kernels of the operators involved in our problem which will play an important role in our Theorem :

Definition 2.5 (Polygon Feynman amplitudes)**.** *Under the geometric setting from paragraph 2.0.4, we set* $\mathbf{G} \in \mathcal{D}'(M \times M, E \boxtimes F^*)$ *to be the Schwartz kernel of* P^{-1} *and* $\mathcal{A} = P + C^\infty(Hom(E,F))$*, where* $P = \Delta, k = 2, E = F$ *in the bosonic case and* $P = D, k = 1, E = E_+, F = E_-$ *in the fermionic case. For every* $n \geqslant 2$*, we formally set*

$$\boxed{t_n = \mathbf{G}(x_1, x_2) \dots \mathbf{G}(x_{n-1}, x_n)\mathbf{G}(x_n, x_1)} \tag{2.4}$$

which is well–defined in $C^\infty\left(M^n \setminus Diagonals, Hom(F,E)^{\boxtimes n}\right)$.

There is a natural fiberwise pairing $\langle t, \varphi\rangle$ *between distributions* t *in* $\mathcal{D}'(M^n, Hom(F,E)^{\boxtimes n})$ *with elements* φ *in* $C^\infty\left(M^n, Hom(E,F)^{\boxtimes n}\right)$ *to get an element* $\langle t, \varphi\rangle \in \mathcal{D}'(M^n)$ *and to get a number, we need to integrate this distribution against a density* $dv \in |\Lambda^{top}|M^n$ *as* $\int_{M^n} \langle t, \varphi\rangle\, dv$.

2.0.6. *Structure of zeta determinants.* Our first main result concerns the mathematical structure of zeta determinants. In the sequel, we denote by $d_n \subset M^n$, the deepest diagonal $\{(x, \dots, x) \in M^n \text{ s.t. } x \in M\} \subset M^n$ and by $N^*(d_n \subset M^n)$ the conormal bundle of d_n. We use the notion of wave front set $WF(t)$ of a distribution t to describe singularities of t in cotangent space and refer to [41, chapter 8] for the precise definitions. For $a \in \mathbb{R}$, we denote by $[a] = \sup_{k \in \mathbb{Z}, k \leqslant a} k$. The bundle of densities on a manifold X will be denoted by $|\Lambda^{top}|X$.

Theorem 1. *The zeta determinants from definition 2.4 extend uniquely as entire functions on* $\mathcal{A}$ *satisfying the factorization formula for* $\|V\|_{C^{d-3}(End(E))}$ *(resp* $\|A\|_{C^{d-1}(Hom(E,F))}$*) small enough :*

$$det_\zeta(\Delta + V) = e^{Q(V)} \exp\left(\sum_{n > \frac{d}{2}} \frac{(-1)^{n+1}}{n} Tr_{L^2}\left((\Delta^{-1}V)^n\right)\right), \text{ in bosonic case} \tag{2.5}$$

$$det_\zeta(D^*(D+A)) = e^{Q(A)} \exp\left(\sum_{n > d} \frac{(-1)^{n+1}}{n} Tr_{L^2}\left((D^{-1}A)^n\right)\right), \text{ in fermionic case} \tag{2.6}$$

where

$$Q(V) = \int_M \langle \ell, V\rangle\, dv + \sum_{2 \leqslant n \leqslant [\frac{d}{2}]} \frac{(-1)^{n+1}}{n} \int_{M^n} \left\langle \mathcal{R}t_n, \mathcal{V}^{\boxtimes n}\right\rangle dv_n \tag{2.7}$$

$$Q(A) = \int_M \langle \ell, A\rangle\, dv + \sum_{2 \leqslant n \leqslant d} \frac{(-1)^{n+1}}{n} \int_{M^n} \left\langle \mathcal{R}t_n, A^{\boxtimes n}\right\rangle dv_n \tag{2.8}$$

- *for* $\ell \in C^\infty(End(E))$ *in bosonic case and* $\ell \in C^\infty(Hom(E_+, E_-))$ *in fermionic case,* $dv \in |\Lambda^{top}|M$*,* $dv_n \in |\Lambda^{top}|M^n$ *are the canonical Riemannian densities*

- *$\mathcal{R}t_n$ is a **distribution of order** m **on** M^n **extending** the distributional product t_n which is well–defined on $M^n \setminus d_n$, $m = d-3$ in the bosonic case and $m = d-1$ in the fermionic case,*
- *the wave front set of $\mathcal{R}t_n$ satisfies the bound*

$$WF(\mathcal{R}t_n) \cap T^\bullet_{d_n} M^n \subset N^*(d_n \subset M^n).$$

There are several consequences of the above result. The definition of Gohberg–Krein's regularized determinants $\det_p, p \in \mathbb{N}$, which are natural generalizations of Fredholm determinants for operators in the Schatten class, is recalled in subsection 3.3. In the following corollary, we use the notion of order of an entire function and discuss its zeroes. An analytic function $F : \mathcal{A} = P + C^\infty(Hom(E,F)) \mapsto \mathbb{C}$ is said to vanish on the subset of noninvertible elements in the sense that for every $\mathcal{V} \in C^\infty(Hom(E,F))$, the entire function $z \mapsto F(P + z\mathcal{V})$ has **divisor** $\{(z, m_z) | \ker(P + z\mathcal{V}) \neq \{0\}, m_z = \dim\ker(P + z\mathcal{V})\}$ and the **order** of $\mathcal{V} \in C^\infty(Hom(E,F)) \mapsto F(P + \mathcal{V})$ equals $\rho(F)$ in the sense the following bound is satisfied : $|F(P+\mathcal{V})| \leqslant Ce^{K\|\mathcal{V}\|^{\rho(F)}_{C^m}}$ for some continuous norm $\|.\|_{C^m}$ on $C^\infty(Hom(E,F))$ and $C, K > 0$ independent of $\mathcal{V}$.

Corollary 2.6 (Zeta determinant for non smooth, non selfadjoint perturbations)**.** *In the notations of the above Theorem, zeta determinants are **entire functions of finite order** $p = [\frac{d}{2}] + 1$ in the bosonic case and $p = d+1$ in the fermionic case on $\mathcal{A}$ vanishing exactly over non invertible elements.*

*They extend as entire functions on **non smooth, non self–adjoint** perturbations*

- *of Δ of regularity $C^{d-3}(End(E)) \cap L^\infty(End(E))$ in the bosonic case,*
- *of D of regularity $C^{d-1}(Hom(E_+, E_-))$ in the fermionic case.*

They are related to Gohberg–Krein's regularized determinants by the following factorization formulas :

$$det_\zeta(\Delta + V) = e^{Q(V)} det_p\left(I + \Delta^{-1}V\right), p = [\frac{d}{2}] + 1 \text{ in bosonic case} \quad (2.9)$$

$$det_\zeta(D^*(D + A)) = e^{Q(A)} det_p\left(I + D^{-1}A\right), p = d + 1 \text{ in fermionic case} \quad (2.10)$$

where Q are the polynomials defined in Theorem 1.

2.0.7. *Finding good determinants.* We next formulate the general problem of finding renormalized determinants with functional properties closed to zeta determinants :

Problem 2.7 (Renormalized determinants)**.** *Under the geometric setting from paragraph 2.0.4, set $\mathcal{A} = P + C^\infty(Hom(E,F)), p = \deg(P)$ where $P = \Delta, p = 2, E = F$ in the bosonic case and $P = D, p = 1, E = E_+, F = E_-$ in the fermionic case. An analytic function $\mathcal{R}\det : \mathcal{A} \mapsto \mathbb{C}$ will be called renormalized determinant if*

(1) *$\mathcal{R}\det$ vanishes exactly on the subset of noninvertible elements and the **order** of $\mathcal{V} \in C^\infty(Hom(E,F)) \mapsto \mathcal{R}\det(P + \mathcal{V})$ equals $[\frac{d}{p}] + 1$, in particular it satisfies the bound :*

$$\boxed{|\mathcal{R}\det(P + \mathcal{V})| \leqslant Ce^{K\|\mathcal{V}\|^{[\frac{d}{p}]+1}_{C^m}}} \quad (2.11)$$

for the continuous norm $\|.\|_{C^m}$ *on* $C^\infty(Hom(E,F))$ *where* $m = d-3$ *in the bosonic case,* $m = d-1$ *in the fermionic case and* $C, K > 0$ *independent of* $\mathcal{V}$*.*

(2) *For* $n > [\frac{d}{p}]$,

$$\boxed{\frac{(-1)^{n-1}}{n-1!}\left(\frac{d}{dz}\right)^n \log \mathcal{R}\det(P+z\mathcal{V})|_{z=0} = Tr_{L^2}\left(\left(P^{-1}\mathcal{V}\right)^n\right).} \tag{2.12}$$

(3) *We further impose a condition of microlocal nature on the second derivative of* $\mathcal{R}\det$ *:*

$$\boxed{\delta_{V_1}\delta_{V_2}\log\mathcal{R}\det\left(P+\mathcal{V}\right) = Tr_{L^2}\Big(\left(P+\mathcal{V}\right)^{-1}V_1\left(P+\mathcal{V}\right)^{-1}V_2\Big)} \tag{2.13}$$

if $supp(V_1)\cap supp(V_2) = \emptyset$ *where the* L^2 *trace is well–defined.*
Under the identification of the second derivative $\delta^2\log\mathcal{R}\det$ *with distributions in* $\mathcal{D}'(M^2, Hom(E,F)\boxtimes Hom(E,F))$ *for* $\|\mathcal{V}\|_{C^m}$ *small enough :*

$$\boxed{WF\left(\delta^2\log\mathcal{R}\det\left(P+\mathcal{V}\right)\right)\cap T^\bullet_{d_2}M^2 \subset N^*\left(d_2\subset M^2\right).} \tag{2.14}$$

Let us motivate the axioms from definition 2.7. Theorem 2 will show that these conditions are optimal to describe all renormalized determinants which can be obtained by local renormalization. It is natural to require our determinants to vanish on noninvertible elements since they generalize the usual Fredholm determinant. We want to minimize the complexity of the entire function $z\mapsto\mathcal{R}\det(P+z\mathcal{V})$ hence its order. We will see in corollary 3.1 that our condition on the order of $\mathcal{R}\det$ is optimal and this is in some sense responsible for the polynomial ambiguity conjectured by Quillen.

Gohberg–Krein's determinants $\det_p$ vanish exactly on non invertible elements and have smallest possible order, but they fail to satisfy the conditions on the second derivative of definition 2.7, hence by our main Theorem 2 they cannot be obtained from renormalization by subtraction of local counterterms. Equations (2.12) and (2.13) are very natural since they are reminiscent of the finite dimensional case. In the seminal work of Kontsevich–Vishik [45, equation (1.4) p. 4], they attribute to Witten the observation that for the zeta determinant, the following identity $\delta_1\delta_2\log\det_\zeta(A) = Tr_{L^2}\left(\delta_1 AA^{-1}\delta_2AA^{-1}\right)$ holds true where δ_1A, δ_2A are pseudodifferential deformations with **disjoint support**. This is not surprising provided we want our determinants to give rigorous meaning to QFT functional integrals. [6] We want to subtract only **smooth local counterterms in** $\mathcal{V}$, this smoothness is imposed by the conditions on the wave front set. The bound on m is also optimal, locality forces renormalized determinants to depend on m-jets of the external potential $\mathcal{V}$.

2.0.8. *Determinants renormalized by subtraction of local counterterms.* In order to give a precise definition of locality, we recall the definition of smooth local functionals.

Definition 2.8 (Local polynomial functionals)**.** *A map* $P : V\in C^\infty(Hom(E,F))\mapsto P(V)\in\mathbb{C}$ *is called local polynomial functional if* P *is smooth in the Fréchet sense and there exists* $k\in\mathbb{N}$, $\Lambda : V\in C^\infty(Hom(E,F))\longmapsto\Lambda\left(j^kV\right)\in C^\infty(M)\otimes_{C^\infty(M)}|\Lambda^{top}|M$ *s.t. for all*

[6] In the present paper, we take this as axiom of our renormalized determinants and the identity 2.13 follows from a formal applications of Feynman rules.

$x \in M$, $\Lambda(j^kV)(x)$ depends polynomially on k-jets of V at x and $P(V) = \int_M \Lambda(j^kV)$. The vector space of local polynomial functionals of degree d depending on the k-jets is denoted by $\mathcal{O}_{loc,d}\left(J^k Hom(E,F)\right)$.

We denote by $\mathbb{C}[\varepsilon^{-\frac{1}{2}}, \log(\varepsilon)]$ the ring of polynomials in $\log(\varepsilon)$ and inverse powers $\varepsilon^{-\frac{1}{2}}$. With the above notion of local functionals one could try to renormalize determinants as follows. If we perturbed the elliptic operator P by a smoothing operator $\mathcal{V} \in \Psi^{-\infty}$ then the Fredholm determinant $\mathcal{V} \mapsto \det_F(I + P^{-1}\mathcal{V})$ is a natural entire function on $\mathcal{A}$ vanishing over non invertible elements. However, $\mathcal{V} \in C^\infty(M, Hom(E,F))$ has only order 0 hence we would like to consider some family $(\mathcal{V}_\varepsilon)_\varepsilon$ of smoothing operators approximating $\mathcal{V}$ and find some family of local polynomial functionals $P_\varepsilon = \int_M \Lambda_\varepsilon(.) \in \mathcal{O}_{loc,d}\left(J^k Hom(E,F)\right) \otimes_{\mathbb{C}} \mathbb{C}[\varepsilon^{-\frac{1}{2}}, \log(\varepsilon)]$ such that the limit $\lim_{\varepsilon\to 0^+} \exp\left(-\int_M \Lambda_\varepsilon(j^k\mathcal{V}(x))\right) \det_F\left(I + P^{-1}\mathcal{V}_\varepsilon\right)$ makes sense where $\det_F\left(I + P^{-1}\mathcal{V}_\varepsilon\right)$ is well defined for $\varepsilon > 0$ since $P^{-1}\mathcal{V}_\varepsilon \in \Psi^{-\infty}$. This operation is called **renormalization by subtraction of local counterterms**.

2.0.9. *Solution of problem 2.7.* We now state the main Theorem of the present paper answering Problem 2.7, the assumptions are from paragraph 2.0.4 and the distributions t_n are from definition 2.5 :

Theorem 2 (Solution of the analytical problem)**.** *A map $\mathcal{R}\det : \mathcal{A} \mapsto \mathbb{C}$ is a solution of problem 2.7 if and only if the following equivalent conditions are satisfied :*

(1) *there exists $Q \in \mathcal{O}_{loc,[\frac{d}{p}]}\left(J^m Hom(E,F)\right)$ such that*

$$\begin{aligned} V \mapsto \mathcal{R}\det(\Delta + V) &= \exp\left(Q(V)\right) det_\zeta\left(\Delta + V\right), p = 2, m = d-3 \text{ for bosons} && (2.15)\\ A \mapsto \mathcal{R}\det(D + A) &= \exp\left(Q(A)\right) det_\zeta\left(D^*(D + A)\right), p = 1, m = d-1 \text{ for fermions.} && (2.16)\end{aligned}$$

(2) *for $\mathcal{V}$ small enough, $\mathcal{R}\det(P + \mathcal{V})$ admits the following representation :*

$$\begin{aligned} \mathcal{R}\det\left(P + \mathcal{V}\right) &= C\exp\Big(\int_M \langle \ell, \mathcal{V}\rangle\, dv + \sum_{2\leqslant n\leqslant \frac{d}{p}} \frac{(-1)^{n+1}}{n} \int_{M^n} \left\langle \mathcal{R}t_n, \mathcal{V}^{\boxtimes n}\right\rangle dv_n \\ &+ \sum_{\frac{d}{p}<n} \frac{(-1)^{n+1}}{n} Tr_{L^2}\left((P^{-1}\mathcal{V})^n\right), \qquad (2.17)\end{aligned}$$

*such that $\ell \in C^\infty(Hom(E,F))$, $dv \in |\Lambda^{top}|M, dv_n \in |\Lambda^{top}|M^n$ the canonical Riemannian densities, the distributional product t_n is well–defined on $M^n \setminus d_n$ and $\mathcal{R}t_n$ is a **distribution of order** m **on** M^n **extending** t_n whose wave front set satisfies the bound*

$$WF\left(\mathcal{R}t_n\right) \cap T^\bullet_{d_n} M^n \subset N^*\left(d_n \subset M^n\right).$$

(3) *$\mathcal{R}\det$ is renormalized by subtraction of local counterterms. There exists a generalized Laplacian Δ with heat operator $e^{-t\Delta}$ and a family $Q_\varepsilon \in \mathcal{O}_{loc,[\frac{d}{p}]}\left(J^m Hom(E,F)\right) \otimes_{\mathbb{C}}$*

$\mathbb{C}[\varepsilon^{-\frac{1}{2}}, \log(\varepsilon)]$ *such that* [7]*:*

$$\mathcal{V} \mapsto \mathcal{R}\det(P+\mathcal{V}) = \lim_{\varepsilon\to 0^+} \exp\left(Q_\varepsilon(\mathcal{V})\right) det_F\left(I + e^{-2\varepsilon\Delta}P^{-1}\mathcal{V}\right). \tag{2.18}$$

As immediate corollary of the above, we get that the group $\mathcal{O}_{loc,[\frac{d}{p}]}\left(J^m Hom(E,F)\right)$ of local polynomial functionals acts freely and transitively on the set of renormalized determinants solutions to 2.7 :

$$Q \in \mathcal{O}_{loc,[\frac{d}{p}]}\left(J^m Hom(E,F)\right) \mapsto \exp\left(Q(\mathcal{V})\right)\mathcal{R}\det(P+\mathcal{V}). \tag{2.19}$$

Equation 2.17 shows that zeta regularized determinants, defined by purely spectral condition, admit a position space representation in terms of Feynman amplitudes and that zeta determinants are just a particular case of some infinite dimensional family of renormalized determinants obtained by subtraction of local counterterms.

Corollary 2.9. *In particular under the assumptions of Theorem 2 and using the same notations,* $p = \deg(P)$ *any function* $\mathcal{R}\det(\Delta+V)$ *can be represented as :*

$$\begin{aligned}\mathcal{R}\det(\Delta+V) &= \exp\left(\int_M \langle \ell, V\rangle\, dv + \sum_{2\leqslant n\leqslant \frac{d}{2}} \frac{(-1)^{n+1}}{n}\int_{M^n}\langle \mathcal{R}t_n, V^{\boxtimes n}\rangle\, dv_n\right) det_{[\frac{d}{p}]+1}\left(I + P^{-1}\mathcal{V}\right)\\ &= \exp\left(\int_M \langle \ell, V\rangle\, dv + \sum_{2\leqslant n\leqslant \frac{d}{2}} \frac{(-1)^{n+1}}{n}\int_{M^n}\langle \mathcal{R}t_n, V^{\boxtimes n}\rangle\, dv_n\right)\prod_{n=1}^{\infty} E_{[\frac{d}{p}]}\left(\frac{1}{\lambda_n}\right)\end{aligned}$$

where $\det_{[\frac{d}{p}]+1}$ *is Gohberg–Krein's determinant,* $E_k(z) = (1-z)e^{z+\frac{z^2}{2}+\dots+\frac{z^k}{k}}, k > 0$ *is a Weierstrass factor and the infinite product is over the sequence* $\{\lambda | \dim\ker(\Delta+\lambda V) \neq 0\}$.

2.0.10. *Renormalized determinants and holomorphic trivializations of Quillen's line bundle.* Under the notations from paragraph 1.0.3, we denote by $\mathcal{O}(\mathcal{L})$ the holomorphic sections from $\mathcal{L}$ and by $\mathcal{O}(\mathcal{A})$ holomorphic functions on $\mathcal{A}$.

Theorem 3 (Holomorphic trivializations and flat connection)**.** *There is a bijection between the set of renormalized* $\mathcal{R}\det$ *from Theorem 2 and global holomorphic trivialization* $\tau : \mathcal{O}(\mathcal{L}) \mapsto \mathcal{O}(\mathcal{A})$ *of the line bundle* $\mathcal{L} \mapsto \mathcal{A}$ *such that*

$$T \in \mathcal{A} \mapsto \tau\left(\iota^*\underline{\det}(T)\right) = \mathcal{R}\det(T). \tag{2.20}$$

The image of the canonical section $\iota^*\underline{\det}(T)$ *under this trivialization being exactly the entire function* $\mathcal{R}\det$ *vanishing over non invertible elements in* $\mathcal{A}$*.*

For every pair (τ_1,τ_2)*, there exists an element* Λ *of the additive group* $\left(\mathcal{O}_{loc,[\frac{d}{p}]}, +\right)$ *s.t.*

$$\tau_1(P+\mathcal{V}) = \exp\left(\int_M \Lambda(j^m\mathcal{V}(x))\right)\tau_2(P+\mathcal{V}), \tag{2.21}$$

[7]the choice of mollifier $e^{-2\varepsilon\Delta}$ is consistent with the GFF interpretation since the covariance of the heat regularized GFF $e^{-\varepsilon\Delta}\phi$ is $e^{-2\varepsilon\Delta}\Delta^{-1}$

for $m = d-3$ (resp $m = d-1$) in the bosonic (resp fermionic) case. For every choice of renormalized $\mathcal{R}\det$, the section $\sigma = \mathcal{R}\det^{-1}\iota^*\underline{\det}$ defines a nowhere vanishing global holomorphic section with canonical holomorphic flat connection ∇ s.t. $\nabla\sigma = 0$.

The ambiguity group that relates all solutions of problem 2.7 is the renormalization group of Stüeckelberg–Petermann as described by Bogoliubov–Shirkov [4] and is interpreted here as a gauge group of the line bundle $\mathcal{L} \mapsto \mathcal{A}$. Our result is a variant of the so called main Theorem of renormalization by Popineau–Stora [55] and studied under several aspects by Brunetti–Fredenhagen [7] and Hollands–Wald [39, 40, 44]. In the aQFT community, there are various recent works exploring the renormalized Wick powers using Euclidean versions of the Epstein–Glaser renormalization [16, 17].

Relation with other works. The way we treat the problem of subtraction of local counterterms is strongly inspired by Costello's work [11] and the point of view of perturbative algebraic quantum field theory which is explained in Rejzner's book [59].

Perrot's notes [52] and Singer's paper [68] on quantum anomalies, which played an important role in our understanding of the topic, are in the real setting. The gauge potential A which is used to perturb the half–Dirac operator preserves the Hermitian structure whereas we do not impose this requirement and view our perturbations as a complex space instead. Actually, our motivation to consider holomorphic determinants in some complexified setting bears strong inspiration from the work of Burghelea–Haller [8, 9] and Braverman–Kappeler [5] on finding some complex valued holomorphic version of the Ray–Singer analytic torsion.

Finally, in a nice recent paper [25], Friedlander generalized the classical multiplicative formula $\det_\zeta(\Delta(I+T)) = \det_\zeta(\Delta)\det_F(I+T)$ when T is smoothing, in [25, Theorem p. 4] connecting zeta determinants, Gohberg–Krein's determinants and Wodzicki residues. This bears a strong similarity with our Corollary 2.6 although our point of view stresses the relation with distributional extensions of products of Green functions[8] in configuration space. Another difference with his work is that we bound the wave front of the functional derivatives of zeta determinants which is important from the QFT viewpoint and is related to the μlocal spectrum condition used in QFT.

Acknowledgements. We warmly thank C. Brouder, Y. Chaubet, Y. Colin de Verdière, C. Dappiaggi, J. Dereziński, K. Gawędzki, C. Guillarmou, A. Karlsson, J. Kellendonk, M. Puchol, G. Rivière, S. Paycha, K. Rejzner, M. Wrochna, B. Zhang for many interesting discussions, questions, advices, remarks that helped me improve my work. We particularly thank D. Perrot for many discussions on anomalies, determinants and Epstein–Glaser renormalization and the Clay foundation for making available the scanned notes of Quillen's notebook where we could find an incredible source of inspiration. I also would like to thank my wife Tho for excellent working conditions at home and a lot of positive motivation for me.

[8]called *Feynman amplitudes* in physics litterature

2.1. **Notations.** dv is used for a smooth density $|\Lambda^{top}|M$ on M. In the sequel, for every pair (B_1, B_2) of Banach spaces, $\mathcal{B}(B_1, B_2)$ denotes the Banach space of bounded operators from $B_1 \mapsto B_2$ endowed with the norm $\|.\|_{\mathcal{B}(B_1,B_2)}$. For any vector bundle E on M, we denote by $\Psi^\bullet(M, E)$ the algebra of pseudodifferential operators on the manifold M acting on sections of the bundle E and when there is no ambiguity we will sometimes use the short notation $\Psi(M)$, $C^k(E), k \in \mathbb{N}$ denotes continuous sections of E of regularity C^k, $H^s(E), s \in \mathbb{R}$ denotes Sobolev sections of E endowed with the norm $\|.\|_{H^s(E)}$ that we shall sometimes write $H^s, \|.\|_{H^s}$ for simplicity[9] and finally $L^p(E), 1 \leqslant p \leqslant +\infty$ denotes L^p sections of E endowed with the norm $\|.\|_{L^p(E)}$. $C^\infty_c(U, E)$ are smooth sections of E compactly supported on U.

For any pair (E, F) of bundles over M, for C^m Schwartz kernels K of operators from $C^m(E) \mapsto C^m(F)$ which are elements of $C^m(M \times M, F \boxtimes E^*)$, we denote by $\|K\|_{C^m(M\times M)}$ their C^m norm which is not to be confused with the operator norm $\|K\|_{\mathcal{B}(C^m(E),C^m(F))}$.

For any Hilbert space H, we denote by $\mathcal{I}_p \subset \mathcal{B}(H, H)$ the Schatten ideal of compact operators whose p-th power is trace class endowed with the norm $\|.\|_p$ defined as $\|A\|_p = \sum_{\lambda\in\sigma(A)} |\lambda|^p$ where the sum runs over the singular values of A.

3. Functional determinants as entire functions with given zeros.

The goal of this section is to introduce the necessary material to prove Theorem 1. We begin with some classical results on entire functions and their zeroes, then we recall some basic results on Fredholm determinants and their generalizations by Gohberg–Krein.

3.1. **Measures of complexity of entire functions.** In this paragraph, we recall some classical invariants of the complexity of entire functions. The **order** $\rho(f) \geqslant 0$ of an entire function f is the infimum of all the real numbers ρ such that for some $A, K > 0$, for all $z \in \mathbb{C}$ $|f(z)| \leqslant Ae^{K|z|^\rho}$. The **critical exponents** of a sequence $|a_n| \to +\infty$, is the infimum of all $\alpha > 0$ such that $\sum_n \frac{1}{|a_n|^\alpha} < +\infty$. Finally the **genus** of f is the order of vanishing of f at $z = 0$. The **divisor** of an entire function f is the set of zeros of f counted with multiplicity. We recall a classical Theorem due to Hadamard on the structure of entire functions with given zeros [60, p. 78–81], [69, Thm 5.1 p. 147] (see also [50, p. 60]):

Theorem 4 (Hadamard's factorization Theorem)**.** *Let $(a_n)_{n\in\mathbb{N}}$ be some sequence such that $\sum_n |a_n|^{-(p+1)} < +\infty$ but $\sum_n |a_n|^{-p} = \infty$. Then any entire function f whose divisor $Z(f) = \{a_n | n \in \mathbb{N}\}$ has order $\rho(f) \geqslant p$, and any entire function s.t. $Z(f) = \{a_n | n \in \mathbb{N}\}$ and $\rho(f) = p$ has a* ***unique representation*** *as :*

$$\boxed{f(z) = z^m e^{P(z)} \prod_{n=1}^{\infty} E_p\left(\frac{z}{a_n}\right)} \tag{3.1}$$

where P is a polynomial of degree p, $E_p(z) = (1 - z)e^{z+\frac{z^2}{2}+\cdots+\frac{z^p}{p}}$ is a Weierstrass factor of order p and m is the genus of f.

[9] when $s < 0$ these are distributional sections

The lower bound on the order of f follows from Jensen's formula. We give a direct application of the above Theorem :

Corollary 3.1. *Under the setting of paragraph 2.0.4, recall $P \in \Psi^k(M;E,F)$ is an invertible elliptic differential operator of order $k>0$. There exists $V \in C^\infty(Hom(E,F))$ such that for any entire function f whose divisor $Z(f)$ corresponds with $\{z|\ker(I+zP^{-1}V)\neq\{0\}\}$, $\rho(f)\geqslant[\frac{d}{k}]+1$ which proves the bound from problem 2.7 is optimal.*

Proof. Since (E,F) are isomorphic bundles, we may choose $V\in C^\infty(Hom(E,F))$ to be a bundle isomorphism hence the corresponding operator is **elliptic** of order 0. Note that $f(z)=0 \implies -z^{-1}\in\sigma\left(P^{-1}V\right)$ where $P^{-1}V\in\Psi^{-k}(M,E)$ by composition hence $P^{-1}V$ is a compact operator. Set $B=V^{-1*}P^*PV^{-1}$. By the results of Seeley [64], the powers $|P^{-1}V|^p=B^{-\frac{p}{2}}$ are elliptic pseudodifferential operators in $\Psi^{kp}(M,E)$. By Hörmander's version of Weyl's law for spectral functions of positive, elliptic pseudodifferential operators [32, Thm 2.1 p. 825], the number of eigenvalues $n_L(B)$ of B less than L grows like a symplectic volume $\int_{\{\sigma(B)(x;\xi)\leqslant L\}\subset T^*M} d^dxd^d\xi \sim_{L\to+\infty} CL^{\frac{d}{2k}}$. The spectral mapping theorem implies eigenvalues of B are of the form $|z|^{-2}$ for $z\in Z(f)$ i.e. eigenvalues of $P^{-1}V$, this implies for $p=[\frac{d}{k}]+1$ that $Tr_{L^2}\left(|P^{-1}V|^p\right)=\sum_{z\in Z(f)}|z|^{-p}<+\infty$ and $Tr_{L^2}\left(|P^{-1}V|^{p-1}\right)=Tr_{L^2}\left(B^{\frac{1-p}{2}}\right)=\sum_{z\in Z(f)}|z|^{-p+1}=+\infty$ hence $\rho(f)\geqslant[\frac{d}{k}]+1$ by Theorem 4. $\square$

Both results show that the solution to the problem of finding entire functions with prescribed zeros is **not unique**, the non unicity is due to the critical exponents of zeros which forces the entire function to have non zero order. So there is an ambiguity relating all possible solutions of the problem which is of the form of exp(Polynomial) by Hadamard's factorization Theorem.

3.2. **An infinite dimensional generalization : Fredholm determinants.** Now we would like to consider the problem of finding entire functions with given zeros in the infinite dimensional setting. But before that, we need to introduce preliminary definitions to deal with entire and analytic functions in infinite dimensional spaces.

3.2.1. *Recollection on holomorphic functions on Fréchet spaces.* In the present paper, we always work with Fréchet spaces of smooth sections of finite rank vector bundles over some compact manifold M. First, let us define what we mean by an entire function on a Fréchet space. For this we start by recalling the definition of finitely analytic (also called Gâteau–holomorphic) functions which is the weakest notion of holomorphicity in ∞–dimension [22, p. 54 def 2.2] :

Definition 3.2 (Finitely analytic functions)**.** *Let Ω open in some Fréchet space E over $\mathbb{C}$. A function $f:E\mapsto\mathbb{C}$ is said to be finitely analytic on Ω if for all $A\in\Omega$, every $B\in E$, $z\in\mathbb{C}\mapsto f(A+zB)$ is a* ***holomorphic germ*** *at $z=0$.*

Beware that finitely analytic maps are not necessarily continuous since any $\mathbb{C}$-linear map $F:E\mapsto\mathbb{C}$ which is not even continuous is always finitely analytic. In the present paper, smooth functions on Fréchet spaces will be understood in the sense of Bastiani [6, Def II.12] as popularized by Hamilton [38] in the context of Fréchet spaces and Milnor [71] which means smooth functions are infinitely differentiable in the sense of Fréchet and all the derivatives $D^nF : U \times E^n \mapsto \mathbb{C}$ are jointly continuous on $U \times E^n$. We recall the notions of functional derivatives since these will play a central role in our approach :

Definition 3.3 (Functional derivatives)**.** *Let $B \mapsto M$ be some Hermitian vector bundle of finite rank on some smooth closed compact manifold M. For a smooth function $f : V \in C^\infty(M,B) \mapsto f(V) \in \mathbb{C}$ where $C^\infty(M,B)$ is the Fréchet space of smooth sections, the n-th functional derivatives*

$$\delta_{h_1}\dots\delta_{h_n} f(V) = \prod_{i=1}^{n} \frac{d}{dt_i} f(V + t_1h_1 + \cdots + t_nh_n)|_{t_1=\cdots=t_n=0} \tag{3.2}$$

is multilinear continuous in $(h_1,\dots,h_n)$, hence it can be identified by the multilinear Schwartz kernel Theorem [6, lemm III.6] *with some distribution $\delta^n f(V)$ in $\mathcal{D}'(M^n, B^{\boxtimes n})$ s.t.*

$$\langle \delta^n f(V), h_1 \boxtimes \cdots \boxtimes h_n\rangle = \delta_{h_1}\dots\delta_{h_n} f(V) \tag{3.3}$$

is jointly continuous in $(V; h_1,\dots,h_n) \in C^\infty(M,B)^{n+1}$ [6, Thm III.10].

So we may give a definition of analytic functions as follows

Definition 3.4 (Analytic functions on Fréchet spaces)**.** *Let $\Omega \subset E$ be some open subset in a Fréchet space E. A function $F : \Omega \subset E \mapsto \mathbb{C}$ is analytic if it is **smooth in the Fréchet sense** and for every $V_0 \in \Omega$, the Taylor series of F converges in some neighborhood of V_0* [10] *and F coincides with its Taylor series : $F(V_0 + h) = \sum_{n=0}^{\infty} F_V^{(n)}(h,\dots,h)$.*

This notion of analyticity is the strongest possible and Fréchet analytic functions are automatically smooth hence C^0 unlike finitely analytic functions. Our goal in this part is to recall the proof that finitely analytic maps near A which are **locally bounded** are analytic near A.

Definition 3.5 (Local boundedness)**.** *A map f is locally bounded near A if there is an open neighborhood $U \subset E$ of A and $0 \leqslant M < +\infty$ such that $|f|_U| \leqslant M$.*

The proof is inspired from the thesis of Douady [23, Prop 2 p. 9] and also [22, p. 57–58].

Proposition 3.6. *Let E be a Fréchet space and $F : E \mapsto \mathbb{C}$ finitely analytic on $\Omega \subset E$. If F is locally bounded at A, in particular if on a ball $B(A,r) = \{B \text{ s.t. } \|B - A\| < r\}$ for a continuous norm $\|.\|$ on E, $\sup_{B(A,r)} |F| \leqslant M < +\infty$, then F is Fréchet differentiable at A at any order and can be identified with its Taylor series near a :*

$$F(A + H) = \sum_{n=0}^{\infty} P_n(H)$$

where each P_n is a continuous polynomial map homogeneous of degree n, the P_n are uniquely determined by $P_n(h) = \frac{n!}{2i\pi}\int_\gamma F(A + \lambda h)\frac{d\lambda}{\lambda^{n=1}}$ and $\sum \|P_n\|\tilde{r}^n < +\infty$ for every $0 < \tilde{r} < r$.

In particular F smooth in some neighborhood of A.

[10] for the Fréchet topology

Proof. This proposition is well–known when B has finite dimension and the expansion $F(a+h)=\sum_n P_n(h)$ is given by the formula $P_n(h)=\frac{1}{2\pi}\int_0^{2\pi}F(a+e^{i\theta}h)e^{-in\theta}d\theta$ where $|h|\leqslant r$ and then we keep this formula in the infinite dimensional case. The integral is that of a continuous function (by finite analyticity) hence is well–defined. If F is bounded by M on a ball of radius $r>0$ for the continuous norm $\|.\|$ then so is P_n. To show that P_n is a homogeneous monomial, we follow Douady's approach by setting $\tilde{P}_n(h_1,\dots,h_n)=\frac{1}{n!}\Delta_{h_1}\dots\Delta_{h_n}P_n$ where Δ_h is the finite difference operator $\Delta_hP(x)=\frac{1}{2}\left(P(x+h)-P(x-h)\right)$. In the finite dimensional case $\tilde{P}_n$ is multilinear and it is the same in the infinite dimensional case since it only depends on the restriction of P_n to some finite dimensional subspace of E. Hence $P_n(h)=\tilde{P}_n(h,\dots,h)$ for some symmetric multilinear map $\tilde{P}_n$. From Cauchy's integral formula, we know that every P_n is bounded by M when $\|h\|\leqslant r$ which implies that $|\tilde{P}_n(h_1,\dots,h_n)|\leqslant\frac{n^nM}{n!}\|h_1\|\dots\|h_n\|$ hence $\tilde{P}_n$ is continuous. From this it results that the series $\sum_n P_n$ has normal convergence and the proposition is proved. □

3.2.2. *Spectral identities related to Fredholm determinants.* We quickly recall some identities relating the Fredholm determinant $\det_F(I+B)$ for a trace class operator $B:H\mapsto H$ acting on some separable Hilbert space H and functional traces of powers of B. These identities will imply that $\det_F$ is an example of entire function on infinite dimensional space $I+$trace class whose zeroes are exactly the non invertible operators. The Fredholm determinant $\det_F(I+B)$ is defined in [66, equation (3.2) p. 32] as $\det_F(I+zB)=\sum_{k=0}^\infty z^kTr(\Lambda^kB)$ where $\Lambda^kB:\Lambda^kH\mapsto\Lambda^kH$ acting on the fermionic Fock space Λ^kH is trace class. Using the bound $\|\Lambda^kB\|_1\leqslant\frac{\|B\|_1}{k!}$ [66, Lemma 3.3 p. 33], it is immediate that $\det(I+zB)$ is an **entire** function in $z\in\mathbb{C}$ (see also [35, Thm 2.1 p. 26]).

For any compact operator B, we will denote by $(\lambda_k(B))_k$ its eigenvalues counted with multiplicity. By [66, Theorem 3.7], the Fredholm determinant can be identified with a Hadamard product and is related to the functional traces by the following sequence of identities :

$$\det_F(1+zB)=\prod_k(1+z\lambda_k(B))=\underbrace{\exp\left(\sum_{m=1}^\infty(-1)^{m+1}z^mTr(B^m)\right)}\tag{3.4}$$

where the term underbraced involving traces is well–defined only when $|z|\|B\|_1<1$. Note the important fact that $\exp\left(\sum_{m=1}^\infty(-1)^{m+1}z^mTr_{L^2}(B^m)\right)$ which is defined on the disc $\mathbb{D}=\{|z|\|B\|_1<1\}$ has **analytic continuation** as an **entire function** of $z\in\mathbb{C}$ and $B\mapsto\det_F(I+B)$ **is an entire function vanishing when** $I+B$ **is non invertible**.

3.3. **Gohberg–Krein's determinants.** Set $p\in\mathbb{N}$ and let A belong to the Schatten ideal $\mathcal{I}_p\subset\mathcal{B}(H,H)$. Following [66, chapter 9], we consider the operator

$$R_p(A)=[(I+A)\exp(\sum_{n=1}^{p-1}\frac{(-1)^n}{n}A^n)-I]\in\mathcal{I}_1$$

which is trace class by [66, Lemma 9.1 p. 75] since $A\in\mathcal{I}_p$. Then following [66, p. 75], we define the regularized determinant as $\det_p(I+zA)=\det_F(1+R_p(zA))$ where $\det_F$ is the Fredholm determinant. The quantity $\det_p$ is well defined since $B=R_p(A)$ is trace class. We have the following :

Lemma 3.7 (Gohberg–Krein's determinants and functional traces)**.** *For all $A\in\mathcal{I}_p$, Gohberg–Krein's determinant $\det_p(1+zA)$ is an **entire function** in $z\in\mathbb{C}$ and is related to traces $Tr(A^n)$ for $n\geqslant p$ by the following formulas :*

$$\boxed{det_p(1+zA)=\exp\left(\sum_{n=p}^{\infty}\frac{(-1)^{n+1}z^n}{n}Tr(A^n)\right)=\prod_k\left[(1+z\lambda_k(A))\exp\left(\sum_{n=1}^{p-1}\frac{(-1)^n}{n}\lambda_k(A)^n\right)\right]}$$

where the infinite product vanishes exactly when $z\lambda_k(A)=-1$ with multiplicity.

The product $\prod_k\left[(1+z\lambda_k(A))\exp\left(\sum_{n=1}^{p-1}\frac{(-1)^n}{n}\lambda_k(A)^n\right)\right]$ reads $\prod_k E_{p-1}(-z\lambda_k(A))$ where E_{p-1} is the Weierstrass factor. From the above, we deduce :

Proposition 3.8. *Let (M,g) be a closed compact Riemannian manifold of dimension d, (E,F) a pair of isomorphic Hermitian bundles over M and $P:C^\infty(E)\mapsto C^\infty(F)$ an invertible elliptic operator of degree k. For any $\mathcal{V}\in C^\infty(Hom(E,F))$, the series $\sum_{n>\frac{d}{k}}\frac{(-1)^{n+1}}{n}Tr\left(\left(P^{-1}\mathcal{V}\right)^n\right)$ converges absolutely for $\|\mathcal{V}\|_{L^\infty(Hom(E,F))}$ small enough and*

$$\mathcal{V}\mapsto\exp\left(\sum_{n>\frac{d}{k}}\frac{(-1)^{n+1}}{n}Tr\left(\left(P^{-1}\mathcal{V}\right)^n\right)\right)=det_{[\frac{d}{k}]+1}\left(I+P^{-1}\mathcal{V}\right)$$

extends uniquely as an entire function on $C^\infty(Hom(E,F))$.

Proof. Introduce some auxiliary bundle isomorphism $E\mapsto F$ which induces an elliptic invertible operator $U\in\Psi^0(M,E,F):L^2(E)\mapsto L^2(F)$ and $UP^{-1}\in\Psi^{-k}(M,E)$ belongs to the Schatten ideal $\mathcal{I}_{[\frac{d}{k}]+1}$ hence $\|UP^{-1}\|_{[\frac{d}{k}]+1}<+\infty$. The claim then follows from Lemma 3.7 applied to $A=P^{-1}\mathcal{V}\in\Psi^{-k}(M,E)$ which belongs to the Schatten ideal $\mathcal{I}_{[\frac{d}{k}]+1}$ and the series converges since the Schatten norm satisfies the estimate :

$$\|P^{-1}\mathcal{V}\|_{[\frac{d}{k}]+1}\leqslant\|U^{-1}\|_{\mathcal{B}(L^2(E),L^2(F))}\|\mathcal{V}\|_{\mathcal{B}(L^2(E),L^2(F))}\|UP^{-1}\|_{[\frac{d}{k}]+1}$$

which can be made <1 if $\|\mathcal{V}\|_{L^\infty(Hom(E,F))}<\|UP^{-1}\|_{[\frac{d}{k}]+1}\|U^{-1}\|^{-1}_{\mathcal{B}(L^2(E),L^2(F))}$. □

4. Proof of Theorem 1.

We work under the setting of paragraph 2.0.4 and the zeta determinants are defined in definitions 2.3 and 2.4. We discuss in great detail the bosonic case for $\det_\zeta(\Delta+V)$ where $V\in C^\infty(M,End(E))$ and we indicate precisely the differences when we deal with the fermionic case for $\det_\zeta(\Delta+D^*A)$ where $\Delta=D^*D$ is a generalized Laplacian, the operator $D:C^\infty(E_+)\mapsto C^\infty(E_-)$ is a generalized Dirac operator and $A\in C^\infty(Hom(E_+,E_-))$. **Both cases consider zeta determinants of a non selfadjoint perturbation of some generalized Laplacian by some differential operator V of order 0 in the bosonic case and $V=D^*A$ of order 1 in the fermionic case.**

4.1. **Perturbations with a gap in the spectrum.** We need to consider small perturbations $\Delta+V$ of some generalized Laplacian Δ s.t. the corresponding heat semigroup $e^{-t(\Delta+V)}$ has exponential decay. In the bosonic case, we assume that Δ is a positive, invertible, symmetric generalized Laplacian hence there is $\delta>0$ such that $\sigma(\Delta)\geqslant\delta$. Therefore there exists some open neighborhood $\mathcal{U}\subset C^\infty(End(E))$ of 0 such that for all small perturbations $V\in\mathcal{U}$, $\Delta+V$ is invertible and $\sigma(\Delta+V)\subset\{Re(z)\geqslant\frac{\delta}{2}\}$. Indeed if $\|V\|_{L^\infty(End(E))}\leqslant\frac{\delta}{2}$ and for $Re(z)<\frac{\delta}{2}-\varepsilon$,

$$\begin{aligned} &\langle u,(\Delta+V-z)u\rangle\geqslant\langle u,\Delta u\rangle-\left(\|V\|_{L^\infty(End(E))}+Re(z)\right)\|u\|^2_{L^2}\\ \geqslant\ &\left(\delta-\|V\|_{L^\infty(End(E))}-Re(z)\right)\|u\|_{L^2}>\varepsilon\|u\|_{L^2}\end{aligned}$$

and the same estimate holds true for the adjoint $\Delta+V^*$.

In the fermionic case the discussion is similar. D invertible implies that $\Delta=D^*D$ satisfies $\sigma(\Delta)\geqslant\delta>0$. If $\|A\|_{L^\infty(Hom(E_+,E_-))}\|\leqslant\frac{\sqrt{\delta}}{2\|\Delta^{-\frac{1}{4}}\|_{\mathcal{B}(L^2,H^{\frac{1}{2}})}\|\Delta^{-\frac{1}{4}}\|_{\mathcal{B}(H^{-\frac{1}{2}},L^2)}\|D^*\|_{\mathcal{B}(H^{\frac{1}{2}},H^{-\frac{1}{2}})}}$ and $Re(z)\leqslant\frac{\delta}{2}-\varepsilon$ then :

$$\begin{aligned} &Re\,\langle u,(\Delta+D^*A-z)\,u\rangle=Re\left\langle\Delta^{\frac{1}{4}}u,\left(\Delta^{\frac{1}{2}}+\Delta^{-\frac{1}{4}}D^*A\Delta^{-\frac{1}{4}}-z\Delta^{-\frac{1}{2}}\right)\Delta^{\frac{1}{4}}u\right\rangle\\ \geqslant\ &\sqrt{\delta}\|u\|^2_{H^{\frac{1}{2}}}-\|\Delta^{-\frac{1}{4}}D^*A\Delta^{-\frac{1}{4}}\|_{\mathcal{B}(H^{\frac{1}{2}},H^{\frac{1}{2}})}\|u\|^2_{H^{\frac{1}{2}}}-Re(z)\delta^{-\frac{1}{2}}\|u\|^2_{H^{\frac{1}{2}}}\\ \geqslant\ &\left(\sqrt{\delta}-\|A\|_{L^\infty(Hom(E_+,E_-))}\|\Delta^{-\frac{1}{4}}\|_{\mathcal{B}(L^2,H^{\frac{1}{2}})}\|\Delta^{-\frac{1}{4}}\|_{\mathcal{B}(H^{-\frac{1}{2}},L^2)}\|D^*\|_{\mathcal{B}(H^{\frac{1}{2}},H^{-\frac{1}{2}})}-\frac{Re(z)}{\sqrt{\delta}}\right)\|u\|^2_{H^{\frac{1}{2}}}\\ \geqslant\ &\frac{\varepsilon}{\sqrt{\delta}}\|u\|^2_{H^{\frac{1}{2}}}\end{aligned}$$

where $\Delta^{-\frac{1}{4}}D^*A\Delta^{-\frac{1}{4}}\in\Psi^0(M,E_+)$ is bounded on $H^1(E_+)$ and $\|D^*\|_{\mathcal{B}(H^{\frac{1}{2}},H^{-\frac{1}{2}})}<+\infty$ by the Calderon–Vaillancourt Theorem. It follows that $\sigma(D^*(D+A))\subset\{Re(z)\geqslant\frac{\delta}{2}\}$.

4.2. **Analyticity.** The results of the present subsection are general enough to apply in both bosonic and fermionic cases. First, for an analytic family $(V_z)_{z\in\mathbb{C}}$ of differential operators of order 1, let us show the analyticity of the map $z\longmapsto\det_\zeta(\Delta+V_z)$. The differential operator $A_z=\Delta+V_z$ is elliptic of order 2 whose principal symbol $|\xi|^2_{g(x)}$ is positive definite for $\xi\neq0$ but A_z is not necessarily self–adjoint when $z\in\mathbb{C}$ and depends analytically on z. Recall we assumed $\ker(\Delta)=\{0\}$. Set $A_z=\Delta+V_z$, then the fundamental solution of the heat operator e^{-tA_z} is defined by [33, Lemma 1.5.7 p. 64]. A well–known consequence of this result is that $z\mapsto Tr_{L^2}(A_z^{-s}),Re(s)>\frac{d}{2}$ [33, Thm 1.12.2] has analytic continuation as meromorphic function in s **without poles at** $s=0$, it is analytic in z near every z_0 s.t. $\Delta+z_0V$ **invertible** and when s is away from a discrete set of poles, in particular $Tr_{L^2}(A_z^{-s})$ is analytic in z near $s=0$.

In particular, the zeta determinant which is defined for invertible A_z as $z\mapsto\det_\zeta(A_z)=\exp\left(\frac{d}{ds}|_{s=0}Tr(A_z^{-s})\right)$ [33, Lemma 1.12.1] is analytic near $z=0$. This shows that $z\mapsto\det_\zeta(\Delta+V_z)$ is holomorphic for every holomorphic family $z\mapsto V_z\in C^\infty(M,End(E))$. Then combining the Hadamard parametrix construction for the heat kernel and Volterra calculus, we next prove that both $V\in C^\infty(End(E))\mapsto\det_\zeta(\Delta+V)$ for bosons and $A\in$

$C^\infty(Hom(E_+,E_-)) \mapsto \det_\zeta(\Delta + D^*A)$ for fermions are **locally bounded** near $V=0$ and $A=0$ respectively which implies the Fréchet analyticity near the origin of both maps by Proposition 3.6.

4.2.1. *Local boundedness and Fréchet analyticity for small V.* The **semigroup** $e^{-t(\Delta+V)}$ is well–defined by the resolvent construction [33] or by the construction of [2]. The treatment of the bosonic and fermionic cases are similar except the perturbation V has order 0 in the bosonic case, it is a potential term in $C^\infty(End(E))$ and $V = D^*A$ is a differential operator of order 1 in the fermionic case and we are interested in the dependence in $A \in C^\infty(Hom(E_+,E_-))$ in the fermionic case.

Proposition 4.1. *For every integer $N > \frac{d}{2}+1$, there exists $\varepsilon, M > 0$ such that*

$$\|V\|_{C^{2N}} \leqslant \varepsilon \implies |det_\zeta(\Delta+V)| \leqslant M \text{ in the bosonic case,}$$
$$\|A\|_{C^{2N+1}} \leqslant \varepsilon \implies |det_\zeta(\Delta+D^*A)| \leqslant M \text{ in the fermionic case.}$$

Therefore the maps $V \mapsto \det_\zeta(\Delta+V)$ and $A \mapsto \det_\zeta(\Delta+D^*A)$ are Fréchet analytic in the sense of definition 3.4 respectively for V and A close enough to 0.

Proof. For simplicity we give the proof in the bosonic case, the fermionic situation is almost verbatim the same using Lemma 4.2 . By the Hadamard–Schwinger–Fock formula, for $Re(s) > \frac{d}{2}$, we may relate the trace of complex powers $Tr\left(A^{-s}\right)$ to heat traces via the Mellin transform as follows :

$$Tr\left(A^{-s}\right) = \frac{1}{\Gamma(s)} \int_0^{+\infty} Tr\left(e^{-t(\Delta+V)}\right) t^{s-1} dt.$$

As we recall in Lemma 4.2, the heat kernel has an asymptotic expansion on the diagonal:

$$e^{-t(\Delta+V)}(x,x) = \frac{1}{(4\pi t)^{\frac{d}{2}}} \left(\sum_{k=0}^{N-1} t^k a_k(x,x;V) \right) + R_N(t,x,x;V)$$

where each $a_k(x,x;V) \in C^\infty(End(E_x))$ is a polynomial homogeneous of degree k on the $(2k-2)$-jets of V and $R_N(t,x,x;V) = \mathcal{O}(t)$ for $N > \frac{d}{2}+1$ and depends analytically on $V \in C^\infty(M)$. The hard part is to control the analyticity of the remainder R_N as a functional of V which is done in Lemma 4.2. Now recall that we choose V is in some neighborhood $\mathcal{U}$ of $V=0$ s.t. $\sigma(\Delta+V) \subset \{Re(z) \geqslant \delta > 0\}, \forall V \in \mathcal{U}$. Using the spectral gap property, we may split the integral $\int_0^{+\infty} Tr\left(e^{-t(\Delta+V)}\right) t^{s-1}dt$ in two parts : $\int_0^{+\infty} Tr\left(Ve^{-t(\Delta+V)}\right) t^{s-1}dt = \int_0^1 Tr\left(e^{-t(\Delta+V)}\right) t^{s-1}dt + \int_1^{+\infty} Tr\left(e^{-t(\Delta+V)}\right) t^{s-1}dt$. We have the upper bound for $t \geqslant 1$,

$$|e^{-t(\Delta+V)}(x,y)| \leqslant \sup_{x\in M} \|e^{-\frac{1}{4}(\Delta+V)}\delta_x\|_{L^2}^2 \|e^{-(t-\frac{1}{2})(\Delta+V)}\|_{\mathcal{B}(L^2,L^2)} \leqslant e^{-(t-\frac{1}{2})\frac{\delta}{2}} \sup_{x\in M} \|e^{-\frac{1}{4}(\Delta+V)}\delta_x\|_{L^2}^2$$

where $\sup_{x\in M} \|e^{-\frac{1}{4}(\Delta+V)}\delta_x\|_{L^2}^2$ is locally bounded in V as proved in Lemma 4.2. This shows immediately that the integral $\int_1^{+\infty} Tr\left(e^{-t(\Delta+V)}\right) t^{s-1}dt$ depends holomorphically on $s \in \mathbb{C}$ uniformly in $V \in \mathcal{U}$. Therefore the heat kernel expansion implies that

$$Tr\left(A^{-s}\right) = \sum_{0 \leqslant k \leqslant \frac{d}{2}} \frac{1}{\Gamma(s)} \frac{\left(\int_{x\in M} a_k(x,x;V)dv(x)\right)}{(4\pi)^{\frac{d}{2}}(s+k-\frac{d}{2})} + \text{holomorphic at } s=0$$

where the holomorphic part is uniformly bounded in $V \in C^\infty(M)$ by the bounds from Lemma 4.2 on the remainder R_N. This implies that $\log\det_\zeta(\Delta+V)$ decomposes as

$$\sum_{1\leqslant k\leqslant N-1, k\neq \frac{d}{2}} \frac{\left(\int_{x\in M} a_k(x,x;V)dv(x)\right)}{(4\pi)^{\frac{d}{2}}(k-\frac{d}{2})} + \int_0^1 \left(\int_M R_N(t,x,x;V)dv(x)\right) t^{-1}dt + \int_1^\infty Tr\left(e^{-t(\Delta+V)}\right)t^{-1}dt$$

where $R_N(t,x,y;V) = \mathcal{O}(t^{N-\frac{d}{2}-\frac{m}{2}})$ is bounded when $\|V\|_{C^{2N}} \leqslant M < +\infty$ for any $N > \frac{d}{2}+1$, thus $V \mapsto \int_0^1 \left(\int_M R_N(t,x,x;V)dv(x)\right) t^{-1}dt$ is locally bounded and $V \mapsto \log\det_\zeta(\Delta+V)$ is Fréchet analytic in V near $V=0$ and its boundedness is controlled by $\|V\|_{C^{2N}}$ for all $N > \frac{d}{2}+1$. □

Lemma 4.2. *We use the notations from paragraph 2.0.4. Let (M,g) be a Riemannian manifold of dimension d and ρ the injectivity radius of (M,g). In the fermionic case $V = D^*A$ and we are interested by the dependence in A. For every $N > \frac{d}{2}+1$ for m some even positive integer, the heat kernel satisfies the following identity :*

$$e^{-t(\Delta+V)}(x,y) = \frac{1}{(4\pi t)^{\frac{d}{2}}}\left(\sum_{k=0}^N \exp\left(-\frac{\mathbf{d}^2(x,y)}{4t}\right)\psi(\mathbf{d}^2(x,y))t^k a_k(x,y;V)\right) + R_N(t,x,y,V)$$

where each $a_k(.,.;V) \in C^\infty(E\boxtimes E^*)$ is a polynomial of degree k on the $(2k-2)$-jets of V (resp $(2k-1)$-jets of A) in the bosonic (resp fermionic) case, $\mathbf{d}^2$ is the square Riemannian distance function, $\psi \in C^\infty_c(\mathbb{R}_{\geqslant 0})$ is a cut–off function satisfying $\psi(u)=1$ when $u \leqslant \frac{\rho^2}{9}$ and $\psi(u)=0$ when $u \geqslant \frac{\rho^2}{4}$. The norm $\|R_N(t,.,.;V)\|_{C^m(M\times M)}$ of the remainder (resp $\|R_N(t,.,.;D^*A)\|_{C^m(M\times M)}$) remains bounded when $\|V\|_{C^{2N}}$ (resp $\|A\|_{C^{2N+1}}$) is bounded.

In particular, the C^m norm of the heat kernel $\|e^{-\frac{1}{2}(\Delta+V)}\|_{C^m(M\times M)}$ is locally bounded in V where the boundedness is controlled by the C^{2N} norm for every $N > \frac{d+m}{2}$ for m some even positive integer.

Proof. We use the existence of an approximate heat kernel $k_N(t,x,y)$ which is supported near the diagonal as follows :

$$k_N(t,x,y) = \sum_{k=0}^N \frac{\exp\left(-\frac{\mathbf{d}^2(x,y)}{4t}\right)}{(4\pi t)^{\frac{d}{2}}}\psi(\mathbf{d}^2(x,y))a_k(x,y;V)t^k$$

where ψ is a cut–off function $\psi=1$ if $t \leqslant \frac{\rho^2}{9}$ and ψ vanishes when $t \geqslant \frac{\rho^2}{4}$ where ρ is the injectivity radius of our Riemannian manifold M. We recall the proof that every $a_k(x,y;V)$ is a polynomial functional in finite jets of the perturbation V as in the proof of [2, Lemma 2.49 p. 98]. In fact the $a_k \in C^\infty(E\boxtimes E^*)$ solve the following hierarchy of transport equations [61, p. 102] [11] :

$$\nabla_{r\frac{d}{dr}}a_k + (k+\frac{r}{4}\frac{d\log(\theta)}{dr})a_k = -(\Delta+V)a_{k-1}, k\geqslant 1, \theta = \det(g_{ij})$$

with $a_0(x,x) = Id_{E_x}$, $r\frac{d}{dr}$ is the radial vector field and every a_k smooth when $x=y$. Immediately this implies that a_0 does not depend on V, a_1 is linear in j^0V in the bosonic

[11] see also [2, Thm 2.26 p. 83] where the equations differ since they work with half–densities

case. In the fermionic case $V = D^*A$, a_0 does not depend on A, a_1 is linear in j^1A, a_2 is polynomial of degree 2 in j^3A. By induction, $a_k, k \geqslant 2$ is a polynomial of degree k on the $(2k-2)$-jets (resp $(2k-1)$-jets) of V (resp A) in the bosonic (resp fermionic) case. It follows that the approximate heat kernel $k_N(t,x,y;V)$ depends polynomially on the $(2N-2)$-jets of V (resp $(2N-1)$-jets of A) of degree N in the bosonic (resp fermionic) case.

By [2, Theorem 2.29], $k_N(t,.,.;V)$ is an approximation of the identity in the sense that for every even $m \in \mathbb{N}$, set $K_N(t)s(x) = \int_{y\in M} k_N(t,x,y;V)s(y)dv(y)$. Then $\|K_N(t)s - s\|_{C^m} \to 0$ when $t \to 0^+$. There exists $\tilde{C}_m, T > 0$ such that $\forall t \in [0,T]$, $\|K_N(t)s\|_{C^m} \leqslant \tilde{C}_m \left(1 + \|V\|_{C^{2N-2}}\right)^N \|s\|_{C^m}$ in the bosonic case and $\|K_N(t)s\|_{C^m} \leqslant \tilde{C}_m \left(1 + \|A\|_{C^{2N-1}}\right)^N \|s\|_{C^m}$ in the fermionic case.

Following [2, p. 77], set $R_t = (\frac{d}{dt} + \Delta + V)K_N(t)$ then we know by the proof of [2, Theorem 2.29] and by construction of k_N that $\|\partial_t^k R_t\|_{C^m(M\times M)} \leqslant C_m \left(1 + \|V\|_{C^{2N}}\right)^{N+1} t^{N-\frac{d}{2}-k-\frac{m}{2}}$ in the bosonic case and $\|\partial_t^k R_t\|_{C^m(M\times M)} \leqslant C_m \left(1 + \|A\|_{C^{2N+1}}\right)^{N+1} t^{N-\frac{d}{2}-k-\frac{m}{2}}$ in the fermionic case for some positive even integer $m \geqslant 2$ and some constant C_m which does not depend on V. Note by construction that R_t is a polynomial on the $2N$-jets of V (resp $(2N+1)$-jets of A) of degree $N+1$ in the bosonic (resp fermionic) case.

Choose $N > \frac{d+m}{2}$ for even $m \geqslant 2$. The definition of the composition $\star$ in the Volterra calculus [36, Def 2.5] reads :

$$A \star B(t,x,y) = \int_0^t \left(\int_{z\in M} A(t-s,x,z)B(s,z,y)dv(z) \right) ds.$$

Using the Volterra calculus, we have the following exact relation between the heat kernel $k(t,x,y;V)$ and the approximate solution $k_N(t,x,y;V)$: $k = \sum_{k=0}^{\infty} (-1)^k k_N \star R^{\star k}$. For every $k \in \mathbb{N}$, each term $k_N \star R^{\star k}$ is a polynomial in the $2N$-jets of V (resp $2N+1$ of A) of degree $k(N+1)+N$. Then by [2, Lemma 2.21] we have the following bound $\|R^{*k+1}(t,.,.)\|_{C^m(M\times M)} \leqslant C_m^{k+1} \left(1 + \|V\|_{C^{2N}}\right)^{(N+1)(k+1)} t^{(k+1)(N-\frac{d}{2})-\frac{m}{2}} \mathrm{Vol}(M)^k \frac{t^k}{k!}$. Combining with the fact that $(K_N(t))_{t\in[0,T]}$ is bounded in $\mathcal{B}\left(C^m(M), C^m(M)\right)$ uniformly in $t \in [0,T]$ for every $m \in \mathbb{N}$, we obtain in the bosonic case :

$$\|K_N \star R^{\star k}\|_{C^m(M\times M)} \leqslant \tilde{C}_m C_m^k \left(1 + \|V\|_{C^{2N}}\right)^{(N+1)(k+1)} t^{k(N-\frac{d}{2})-\frac{m}{2}} \mathrm{Vol}(M)^{k-1} \frac{t^{k-1}}{k-1!},$$

in the fermionic case :

$$\|K_N \star R^{\star k}\|_{C^m(M\times M)} \leqslant \tilde{C}_m C_m^k \left(1 + \|A\|_{C^{2N+1}}\right)^{(N+1)(k+1)} t^{k(N-\frac{d}{2})-\frac{m}{2}} \mathrm{Vol}(M)^{k-1} \frac{t^{k-1}}{k-1!},$$

and therefore the series defining $R_N(t,.,.;V) = \sum_{k=1}^{\infty} K_N \star R^{\star k}$ satisfies :

$$\|R_N(t,.,.;V)\|_{C^m(M\times M)} \quad \leqslant \quad \sum_{k=1}^{\infty} \tilde{C}_m C_m^k \left(1 + \|V\|_{C^{2N}}\right)^{(N+1)(k+1)} t^{k(N-\frac{d}{2})-\frac{m}{2}} \mathrm{Vol}(M)^{k-1} \frac{t^{k-1}}{k-1!}$$

in the bosonic case and

$$\|R_N(t,.,.;D^*A)\|_{C^m(M\times M)} \quad \leqslant \quad \sum_{k=1}^{\infty} \tilde{C}_m C_m^k \left(1 + \|A\|_{C^{2N+1}}\right)^{(N+1)(k+1)} t^{k(N-\frac{d}{2})-\frac{m}{2}} \mathrm{Vol}(M)^{k-1} \frac{t^{k-1}}{k-1!}$$

in the fermionic case where both series converge absolutely in their respective norms. □

4.3. **Functional derivatives.** Inspired by the nice exposition in Chaumard's thesis [10, p. 31-32], we calculate the derivatives in z of $\log\det_\zeta(\Delta+zV)$ near $z=0$ and we find in the bosonic case that for $n>\frac{d}{2}$, the derivatives of order n of $\log\det_\zeta(\Delta+V)$ equals $(-1)^{n-1}(n-1)!Tr_{L^2}\left((\Delta^{-1}V)^n\right)$ where the L^2–trace is well–defined, in the fermionic case a similar result holds true for $n>d$.

We introduce a method which allows to simultaneously calculate the functional derivatives of $\log\det_\zeta$ and bound the wave front set of their Schwartz kernels. For any analytic family $(V_t)_{t\in\mathbb{R}^n}$ of perturbations, setting $A_t=\Delta+V_t$ we know that $Tr(\Delta+V_t)^{-s}$ is **holomorphic** near $s=0$ and depends smoothly on $t=(t_1,\dots,t_n)\in\mathbb{R}^n$, and satisfies the variation formula [33, d) Thm 1.12.2 p. 108] :

$$\boxed{\frac{d}{dt_i}Tr\left(A_t^{-s}\right)=-sTr\left(\frac{dV}{dt_i}A_t^{-s-1}\right),i\in\{1,\dots,n\}} \tag{4.1}$$

which is valid away from the poles of the analytic continuation in s of $Tr_{L^2}\left(A_t^{-s}\right)$ hence the above equation holds true near $s=0$. The holomorphicity of $Tr(\Delta+V_t)^{-s}$ implies the Laurent series expansion $Tr(\Delta+V_t)^{-s}=\sum_{k=0}^\infty a_k(V_t)s^k$ near $s=0$. By definition $\log\det_\zeta(\Delta+V_t)=-\frac{d}{ds}|_{s=0}Tr(\Delta+V_t)^{-s}$ which implies that

$$\frac{d}{dt_i}\log\det_\zeta(\Delta+V_t)=-\frac{d}{dt_i}\frac{d}{ds}|_{s=0}Tr(\Delta+V_t)^{-s}=-\frac{da_1(V_t)}{dt_i}$$

$$=\quad -\frac{d}{ds}|_{s=0}\sum_{k=0}^\infty\frac{da_k(V_t)}{dt_i}s^k=-\frac{d}{ds}|_{s=0}\frac{d}{dt_i}Tr(\Delta+V_t)^{-s}=\frac{d}{ds}|_{s=0}sTr\left(\frac{dV}{dt_i}A_t^{-s-1}\right).$$

Thus for higher derivatives, using equation (4.1), we immediately deduce that

$$\begin{aligned}\frac{d}{dt_1}\dots\frac{d}{dt_{n+1}}\log\det_\zeta(\Delta+V_t) &= \frac{d}{ds}|_{s=0}s\frac{d}{dt_1}\dots\frac{d}{dt_n}Tr\left(\frac{dV_t}{dt_{n+1}}A_t^{-s-1}\right)\\ &= FP|_{s=0}\frac{d}{dt_1}\dots\frac{d}{dt_n}Tr\left(\frac{dV_t}{dt_{n+1}}A_t^{-s-1}\right)\end{aligned}$$

where the finite part FP of a meromorphic germ at $s=0$ is defined to be the constant term in the Laurent series expansion about $s=0$.

So specializing the above identity to the family $t\in\mathbb{R}^n\mapsto V+t_1V_1+\dots+t_{n+1}V_{n+1}$ we find a preliminary formula for the functional derivatives of $\log\det_\zeta$ for bosons :

$$\boxed{\delta_{V_1}\dots\delta_{V_{n+1}}\log\det_\zeta(\Delta+V)=FP|_{s=0}Tr\left(\delta_{V_1}\dots\delta_{V_n}\left(\Delta+V\right)^{-s-1}V_{n+1}\right),} \tag{4.2}$$

and for fermions

$$\boxed{\delta_{A_1}\dots\delta_{A_{n+1}}\log\det_\zeta(\Delta+D^*A_0)=FP|_{s=0}Tr\left(\delta_{A_1}\dots\delta_{A_n}\left(\Delta+D^*A_0\right)^{-s-1}D^*A_{n+1}\right).}$$

At this level of generality the formulas in both bosonic and fermionic cases are very similar just replacing V by D^*A gives the fermionic formulas. To calculate more explicitely the functional derivatives on the r.h.s of equation 4.2, we shall study in more details the analytic map $V\mapsto(\Delta+V)^{-s-1}$ in both bosonic and fermionic situations.

4.3.1. *From the heat operator $e^{-t(\Delta+V)}$ to $(\Delta+V)^{-s-1}$ as analytic functions of V.* Assume Δ is a generalized Laplacian, not necessarily symmetric, s.t. $\sigma(\Delta)\subset\{Re(z)\geqslant\delta>0\}$. By Duhamel formula, the heat operator $e^{-t(\Delta+V)}$ can be expressed in terms of $e^{-t\Delta}$ as the Volterra series :

$$e^{-t(\Delta+V)}=\sum_{k=0}^{\infty}(-1)^k\int_{t\Delta_k}e^{-(t-t_k)\Delta}V\dots Ve^{-t_1\Delta} \tag{4.3}$$

where the series converges absolutely in $\mathcal{B}\left(L^2,L^2\right)$ since in the bosonic case, we have the bound

$$\|\int_{t\Delta_k}e^{-(t-t_k)\Delta}V\dots Ve^{-t_1\Delta}\|_{\mathcal{B}(L^2,L^2)}\leqslant e^{-t\delta}\frac{t^k\|V\|^k_{\mathcal{B}(L^2,L^2)}}{k!}.$$

In the fermionic case, the convergence is slightly more subtle. We start from the bound

Lemma 4.3. *Assume that $\Delta=D^*(D+A_0)\in\mathcal{A}$ is a generalized Laplacian s.t. $\sigma(\Delta)\subset\{Re(z)\geqslant\delta>0\}$. For any differential operator P of order 1,*

$$\|e^{-t\Delta}P\|_{\mathcal{B}(L^2,L^2)}\leqslant Ct^{-\frac{1}{2}}e^{-\frac{t}{2}\delta}. \tag{4.4}$$

Proof. Assume that $\Delta=D^*(D+A_0)\in\mathcal{A}$ is a generalized Laplacian s.t. $\sigma(\Delta)\subset\{Re(z)\geqslant\delta>0\}$ hence Δ is not necessarily self–adjoint. As recalled in Lemma 4.2, the asymptotic expansion of the heat kernel has the form $e^{-t\Delta}(x,y)=\sum_{k=0}^{\frac{d}{2}-1}\frac{a_k(x,y)}{(4\pi)^{\frac{d}{2}}}t^{k-\frac{d}{2}}e^{-\frac{\mathbf{d}^2(x,y)}{4t}}\psi(\mathbf{d}^2(x,y))+R(t,x,y)$ where $\|R(t,x,y)\|_{C^1(M\times M)}=\mathcal{O}(t^{-\frac{1}{2}})$. From the observation that $\|e^{-t\Delta}P\|_{\mathcal{B}(L^2,L^2)}=\|P^*e^{-t\Delta^*}\|_{\mathcal{B}(L^2,L^2)}$ where Δ^* is also a generalized Laplacian satisfying the same assumptions as Δ, it is the same to study $e^{-t\Delta}P$ and $Pe^{-t\Delta}$ for P a first order differential operator. Note that the kernel of $Pe^{-t\Delta}$ is obtained by differentiating the heat kernel. In the calculation, we encounter terms of the form $\partial_{x^i}e^{-\frac{\mathbf{d}^2(x,y)}{4t}}=\mathcal{O}\left(\frac{|x-y|}{2t}e^{-\frac{\mathbf{d}^2(x,y)}{4t}}\right)=\mathcal{O}(t^{-\frac{1}{2}}e^{-\frac{\mathbf{d}^2(x,y)}{4t}})$ using $\mathbf{d}^2(x,y)=g_{\mu\nu}(x^\mu-y^\mu)(x^\nu-y^\nu)+\mathcal{O}\left(|x-y|^3\right)$ by [15, Lemma 5.2 p. 23]. Hence $Pe^{-t\Delta}(x,y)=K(t,x,y)$ has the form

$$K(t,x,y)=t^{-\frac{d+1}{2}}A(t,x,y)+t^{-\frac{1}{2}}B(t,x,y)$$

where $\sup_{t\in[0,1]}\|B(t,.,.)\|_{C^0(M\times M)}<+\infty$ and $|A(t,x,y)|\leqslant P(t)e^{-\frac{\mathbf{d}^2(x,y)}{4t}}$ where P is polynomial in t. Then the key idea is to note that we can localize the study of K on the diagonal. If we let $\chi\in C^\infty(M\times M),0\leqslant\chi\leqslant1$ to be any cut–off function which equals 1 near the diagonal $d_2\subset M\times M$ then by the fast decay off–diagonal $\|K(t,.,.)(1-\chi)\|_{C^0(M\times M)}=\mathcal{O}(t^\infty)$ hence

$$\varphi\in L^2(E)\mapsto\int_{y\in M}K(t,.,y)(1-\chi)\varphi(y)dv(y)\in L^2(E)$$

is bounded uniformly in $t\in[0,1]$. In local coordinates near $(p,p)\in U\times U\subset M\times M$, let $\psi\in C^\infty_c(U\times U,\mathbb{R}_{\geqslant0})$ be a cut–off function where $\operatorname{supp}(\psi)\subset\{|x-y|\leqslant\varepsilon,y\in U\}$. By choosing U small enough near p and by the above discussion, we know that $|K(t,x,y)|\leqslant C_1t^{-\frac{d+1}{2}}e^{-\frac{K|x-y|^2}{t}}+Mt^{-\frac{1}{2}}$ for all $(t,x,y)\in[0,1]\times U\times U$ for some $C_1=\sup_{t\in[0,1]}P(t),M=\sup_{(t,x,y)\in[0,1]\times M^2)}B(t,x,y)$ and $K=\frac{a}{4}$ since the Riemannian distance satisfies the following lower bound w.r.t. the Euclidean distance $a|x-y|^2 \leqslant \mathbf{d}^2(x,y)$ for all $(x,y)\in U\times U$ for some $a>0$ as in the proof of [15, Lemma 5.3]. Note that both :

$$\sup_{y\in\mathbb{R}^d} t^{\frac{1}{2}}\int_{x\in\mathbb{R}^d}|K(t,x,y)|\psi(x,y)d^dx, \text{ and } \sup_{x\in\mathbb{R}^d} t^{\frac{1}{2}}\int_{y\in\mathbb{R}^d}|K(t,x,y)|\psi(x,y)d^dy \tag{4.5}$$

are bounded uniformly in $t\in[0,1]$ since

$$\begin{aligned}&\sup_{y\in U} t^{\frac{1}{2}}\int_{x\in\mathbb{R}^d} t^{-\frac{d+1}{2}}|K(t,x,y)|\psi(x,y)d^dx\\ \leqslant\ &\sup_{y\in U}\left(M\int_{\mathbb{R}^d}\psi(x,y)d^dx+\int_{x\in\mathbb{R}^d}C_1e^{-K|x-y|^2}\psi(\sqrt{t}(x-y)+y,y)d^dx\right)<+\infty\end{aligned}$$

where the r.h.s is bounded uniformly in $t\in[0,1]$. It follows that for every $U\subset M$ and $\psi\in C^\infty_c(U\times U,\mathbb{R}_{\geqslant 0})$ as above, the inequalities 4.5 imply that the family of operators $\varphi\in L^2(E)\mapsto t^{\frac{1}{2}}\int_M K(t,.,y)\psi(.,y)\varphi(y)dv(y)\in L^2(E)$ is bounded uniformly in $t\in[0,1]$ by Schur's test [73, Prop 5.1 p. 573]. Since open subsets of the form $U\times U$ cover the diagonal $d_2\subset M\times M$, we may extract a finite subcover $\cup_{i\in I}U_i\times U_i$ of d_2 and a partition of unity $(\psi_i)_{i\in I}$ subordinated to the subcover. Then the finite sum $t^{\frac{1}{2}}\sum_{i\in I}K(t,.,.)\psi_i(.,.)$ defines a bounded family of operators on $L^2(E)$ which implies for the moment that $\sup_{t\in[0,1]}t^{\frac{1}{2}}\|Pe^{-t\Delta})\|_{\mathcal{B}(L^2,L^2)}\leqslant C_3$. To conclude use the relation $\|e^{-t\Delta}\|_{\mathcal{B}(L^2,L^2)}\leqslant\|e^{-\frac{t}{2}\Delta}\|^2_{\mathcal{B}(L^2,L^2)}\leqslant e^{-\frac{t}{2}\delta}C_3\sqrt{2}t^{-\frac{1}{2}}$ for $t\leqslant 1$ and $\|e^{-t\Delta}\|_{\mathcal{B}(L^2,L^2)}\leqslant e^{-t\delta}$ when $t\geqslant 1$ which implies the claim for some constant C which can be chosen larger than $C_3\sqrt{2}$. □

Therefore setting $V=D^*A$, we find that the series on the r.h.s of identity (4.3) converges absolutely in $\mathcal{B}\left(L^2,L^2\right)$ by the bound

$$\begin{aligned}\|\int_{t\Delta_k}e^{-(t-t_k)\Delta}D^*A\dots D^*Ae^{-t_1\Delta}\|_{\mathcal{B}(L^2,L^2)} &\leqslant \|A\|^k_{\mathcal{B}(L^2(E_+),L^2(E_-))}e^{-\frac{t}{2}\delta}C^k\int_0^t|t-t_k|^{-\frac{1}{2}}\dots\int_0^{t_2}|t_1|^{-\frac{1}{2}}dt_1\dots dt_k\\ &= \|A\|^k_{\mathcal{B}(L^2(E_+),L^2(E_-))}e^{-\frac{t}{2}\delta}\frac{(k+1)C^kt^k\Gamma(\frac{1}{2})^{k+1}}{\Gamma(\frac{k+3}{2})},\end{aligned}$$

where C is the constant of Lemma 4.3 and the r.h.s. is the general term of some convergent series by the asymptotic behaviour of the Euler Γ function [12].

Furthermore using the Hadamard–Fock–Schwinger formula, for $Re(s)>0$, we find that

$$\begin{aligned}(\Delta+V)^{-s-1}V &= \sum_{k=0}^\infty\frac{(-1)^k}{\Gamma(s+1)}\int_0^\infty t^s\int_{t\Delta_k}e^{-(t-t_k)\Delta}V\dots Ve^{-t_1\Delta}Vdt \text{ for bosons,}\\ (\Delta+D^*A)^{-s-1}D^*A &= \sum_{k=0}^\infty\frac{(-1)^k}{\Gamma(s+1)}\int_0^\infty t^s\int_{t\Delta_k}e^{-(t-t_k)\Delta}D^*A\dots D^*Ae^{-t_1\Delta}D^*Adt \text{ for fermions,}\end{aligned}$$

[12] Rewrite $\int_0^t|t-t_k|^{-\frac{1}{2}}\dots\int_0^{t_2}|t_1|^{-\frac{1}{2}}dt_1\dots dt_k=\int_{\{u_1+\dots+u_{k+1}=t\}}\prod_{i=1}^{k+1}u_i^{-\frac{1}{2}}d\sigma$, $d\sigma$ measure on the simplex $\{u_1+\dots+u_{k+1}=t\}$. Note that $\int_{u_1+\dots+u_{k+1}=t}\prod_{i=1}^{k+1}u_i^{-\frac{1}{2}}d\sigma=\frac{d}{dt}\int_{\{u_1+\dots+u_{k+1}\leqslant t\}}\prod_{i=1}^{k+1}u_i^{-\frac{1}{2}}du_i=\frac{d}{dt}\frac{t^{k+1}\Gamma(\frac{1}{2})^{k+1}}{\Gamma(\frac{k+3}{2})}$ following a beautiful identity due to Dirichlet

where both series converge absolutely in $V \in \mathcal{B}\left(L^2(E), L^2(E)\right)$ (resp $A \in \mathcal{B}(L^2(E_+), L^2(E_-))$) by the above bounds since we have exponential decay in t.

From the above we know that

Lemma 4.4 (Functional derivatives of $\log\det_\zeta$)**.** *Following the notations from definitions (2.4) and (2.3). Let M be a smooth, closed, compact Riemannian manifold of dimension d, and Δ (resp $\Delta = D^*D$) some generalized Laplacian acting on E (resp E_+) s.t. $\sigma(\Delta) \subset \{Re(z) \geqslant \delta > 0\}$ for bosons (resp fermions). The functional derivatives of $\log\det_\zeta$ satisfy the following identities. For bosons, for every $(V_1, \dots, V_{k+1}) \in L^\infty(M, End(E))^{k+1}$,*

$$\boxed{\frac{1}{k!}\delta_{V_1}\dots\delta_{V_{k+1}} \log det_\zeta(\Delta + V)|_{V=0} = FP|_{s=0}\frac{(-1)^k}{\Gamma(s+1)}\int_{[0,\infty)^{k+1}}(\sum_{e=1}^{k+1} u_e)^s Tr\left(e^{-u_{k+1}\Delta}V_1\dots e^{-u_1\Delta}V_{k+1}\right)\prod_{e=1}^{k+1} du_e.}$$

For fermions, for every $(A_1, \dots, A_{k+1}) \in L^\infty(M, Hom(E_+, E_-))$

$$\boxed{\frac{1}{k!}\delta_{A_1}\dots\delta_{A_{k+1}} \log det_\zeta(\Delta + D^*A)|_{A=0} = FP|_{s=0}\frac{(-1)^k}{\Gamma(s+1)}\int_{[0,\infty)^{k+1}}(\sum_{e=1}^{k+1} u_e)^s Tr\left(e^{-u_{k+1}\Delta}D^*A_1\dots e^{-u_1\Delta}D^*A_{k+1}\right)\prod_{e=1}^{k+1} du_e.}$$

We want to determine more explicitely the above functional derivatives. Our next task is to make sense and bound integrals of the form

$$Tr(\int_{[0,\infty)^{k+1}}(\sum_{e=1}^{k+1} u_e)^s e^{-u_{k+1}\Delta}V_1\dots e^{-u_1\Delta}V_{k+1}\prod_{e=1}^{k+1} du_e) = I(s, V_1, \dots, V_{k+1})$$

for s near 0.

As in [15], the strategy relies on methods from quantum field theory : using the symmetries of the integrand by permutation of variables, we integrate on a simplex $\{u_{k+1} \geqslant \dots \geqslant u_1 \geqslant 0\}$ called Hepp's sector :

$$\begin{aligned}
&\int_{[0,\infty)^{k+1}}(\sum_{e=1}^{k+1} u_e)^s e^{-u_{k+1}\Delta}V\dots e^{-u_1\Delta}V\prod_{e=1}^{k+1} du_e \\
= \;& (k+1)!\int_{\{u_{k+1}\geqslant\dots\geqslant u_1\geqslant 0\}}(u_1+\dots+u_{k+1})^s e^{-u_{k+1}\Delta}V\dots e^{-u_1\Delta}V du_1\dots du_{k+1}
\end{aligned}$$

We will show that the only divergence is in the variable u_{k+1}. In the next definition, we cut the integral in two parts, $u_{k+1} \geqslant 1$ and $u_{k+1} \leqslant 1$.

Definition 4.5 (Decomposition)**.** *Under the assumptions of Lemma 4.4. We set*

$$\boxed{I(s, V_1, \dots, V_{k+1}) = \int_{[0,\infty)^{k+1}}(\sum_{e=1}^{k+1} u_e)^s Tr\left(e^{-u_{k+1}\Delta}V_1\dots e^{-u_1\Delta}V_{k+1}\right)\prod_{e=1}^{k+1} du_e} \tag{4.6}$$

that we shall decompose in two pieces

$$I(s; V_1, \dots, V_{k+1}) = S(s; V_1, \dots, V_{k+1}) + R(s; V_1, \dots, V_{k+1}) \tag{4.7}$$

where

$$R(s;V_1,\dots,V_{k+1})=(k+1)!\int_{\{u_{k+1}\geqslant\dots\geqslant u_1\geqslant 0,u_{k+1}\geqslant 1\}}(\sum_{e=1}^{k+1}u_e)^sTr\left(e^{-u_{k+1}\Delta}V_1\dots e^{-u_1\Delta}V_{k+1}\right)\prod_{e=1}^{k+1}du_e, \tag{4.8}$$

and

$$S(s;V_1,\dots,V_{k+1})=(k+1)!\int_{\{1\geqslant u_{k+1}\geqslant\dots\geqslant u_1\geqslant 0\}}(\sum_{e=1}^{k+1}u_e)^sTr\left(e^{-u_{k+1}\Delta}V_1\dots e^{-u_1\Delta}V_{k+1}\right)\prod_{e=1}^{k+1}du_e. \tag{4.9}$$

We use the above decomposition for both bosons and fermions where $(V_i=D^*A_i,A_i\in C^\infty(Hom(E_+,E_-)))_{i=1}^{k+1}$ for fermions. The function S (resp R) is the singular (resp regular) part of I. We will later deal with the singular part S using the heat calculus of Melrose [49, Chapter 7] [36, 12]. We shall first show that the regular part R has analytic continuation as holomorphic function in s on the whole complex plane.

Lemma 4.6. *Following the notations from definitions (2.4) and (2.3). Let M be a smooth, closed, compact Riemannian manifold of dimension d, and Δ (resp $\Delta=D^*D$) some generalized Laplacian acting on E (resp E_+) s.t. $\sigma(\Delta)\subset\{Re(z)\geqslant\delta>0\}$ for bosons (resp fermions). For every $(V_1,\dots,V_{k+1})\in C^\infty(End(E))^{k+1}$ in the bosonic case and for every $(A_1,\dots,A_{k+1})\in C^\infty(Hom(E_+,E_-))$ where $V_1=D^*A_1,\dots,V_{k+1}=D^*A_{k+1}$ in the fermionic case, the regular part $R(s;V_1,\dots,V_{k+1})$ has analytic continuation as a holomorphic function of $s\in\mathbb{C}$.*

Proof. For $p>d$ and $B\in\mathcal{B}(L^2,H^p)$, B is trace class and satisfies the simple bound $|Tr_{L^2}(B)|\leqslant C\|B\|_{\mathcal{B}(L^2,H^p)}$ [24, Prop B 20]. Hence in the bosonic case,

$$\begin{aligned}
|Tr_{L^2}\left(e^{-u_{k+1}\Delta}V_1\dots e^{-u_1\Delta}V_{k+1}\right)| &\leqslant \|e^{-\frac{1}{2}\Delta}\|_{\mathcal{B}(L^2,H^p)}\|e^{-(u_{k+1}-\frac{1}{2})\Delta}\|_{\mathcal{B}(L^2,L^2)}\prod_{i=1}^{k+1}\|V_i\|_{\mathcal{B}(L^2,L^2)}\\
&\leqslant e^{-(u_{k+1}-\frac{1}{2})\delta}\|e^{-\frac{1}{2}\Delta}\|_{\mathcal{B}(L^2,H^p)}\prod_{i=1}^{k+1}\|V_i\|_{\mathcal{B}(L^2,L^2)}
\end{aligned}$$

the integrand has exponential decay which ensures the holomorphicity.

In the fermionic case where $(V_i=D^*A_i,A_i\in C^\infty(Hom(E_+,E_-)))_{i=1}^{k+1}$, the bound reads :

$$\begin{aligned}
&|Tr_{L^2}\left(e^{-u_{k+1}\Delta}V_1\dots e^{-u_1\Delta}V_{k+1}\right)|\\
\leqslant\ & \|e^{-\frac{1}{2}\Delta}\|_{\mathcal{B}(L^2,H^p)}\|e^{-(u_{k+1}-\frac{1}{2})\Delta}D^*\|_{\mathcal{B}(L^2,L^2)}\prod_{i=1}^{k}\|e^{-u_i\Delta}D^*\|_{\mathcal{B}(L^2,L^2)}\prod_{i=1}^{k+1}\|A_i\|_{\mathcal{B}(L^2,L^2)}\\
\leqslant\ & \sqrt{2}C^{k+1}\|e^{-\frac{1}{2}\Delta}\|_{\mathcal{B}(L^2,H^p)}e^{-(\frac{u_{k+1}}{2}-\frac{1}{4})\delta}\prod_{i=1}^{k}u_i^{-\frac{1}{2}}\prod_{i=1}^{k+1}\|A_i\|_{\mathcal{B}(L^2,L^2)}
\end{aligned}$$

where C is the constant from Lemma 4.3, $\|e^{-\frac{1}{2}\Delta}\|_{\mathcal{B}(L^2,H^p)}<+\infty$ since the heat kernel is smoothing and the r.h.s has exponential decay in u_{k+1} which ensures holomorphicity. $\square$

It remains to deal with the term $S(s;V_1,\dots,V_{k+1})$ involving the integral for $u_{k+1}\in[0,1]$.

4.3.2. *The tadpole case when $k+1=1$.* In this simple case, for the bosonic case, we directly use the diagonal asymptotic expansion of the heat kernel [33, Lemma 1.8.2] $e^{-t\Delta}(x,x)\sim\sum_{k=0}^{\infty}\frac{a_k(x,x)t^{k-\frac{d}{2}}}{(4\pi)^{\frac{d}{2}}}$ which yields $FP|_{s=0}\frac{1}{\Gamma(s+1)}\int_0^{+\infty}u^sTr\left(e^{-u\Delta}V\right)du=Tr\left(e^{-\Delta}\Delta^{-1}V\right)$ $+\int_0^1 dt\int_M r_{N+1}(t,x,x)V(x)dv+\sum_{k=0,k\neq\frac{d}{2}-1}^{N}\frac{\left(\int_M a_k(x,x)V(x)dv\right)}{(4\pi)^{\frac{d}{2}}\left(s+1+k-\frac{d}{2}\right)}-\frac{\Gamma'(1)}{(4\pi)^{\frac{d}{2}}}\left(\int_M a_{\frac{d}{2}-1}(x,x)V(x)dv\right)$. This means that $FP|_{s=0}\frac{1}{\Gamma(s+1)}\int_0^{+\infty}u^sTr\left(e^{-u\Delta}V\right)du=\int_M\langle\ell,V\rangle dv$ where $\ell\in C^\infty\left(End(E)\right)$ and $dv\in|\Lambda^{top}|M$. A similar result holds true in the fermionic case where we find that $FP|_{s=0}\frac{1}{\Gamma(s+1)}\int_0^{+\infty}u^sTr\left(e^{-u\Delta}D^*A\right)du=\int_M\langle\ell,A\rangle dv$ where $\ell\in C^\infty(Hom(E_-,E_+))$ and $dv\in|\Lambda^{top}|M$.

4.3.3. *When $k+1>\frac{d}{2}$ in the bosonic case and $k+1>d$ in the fermionic case.* We use the formalism of the heat calculus of Melrose as exposed in the work of Grieser [36] (see also [72, p. 62] for related construction) whose notations are adopted. We start from the fact that in local coordinates, $e^{-t\Delta}(x,y)=t^{-\frac{d}{2}}\tilde{A}(t,\frac{x-y}{\sqrt{t}},y)$ where $\tilde{A}\in C^\infty\left([0,\infty)_{\frac{1}{2}}\times\mathbb{R}^d\times U,E\boxtimes E^*\right)$ since the heat kernel is an element in $\Psi_H^{-1}(M,E)$ [36, definition 2.1 p. 6]. Then we note that for $k+1>\frac{d}{2}$, the $k+1$-fold composition $K\star\dots\star K$ belongs to $\Psi_H^{-k-1}(M,E)$ by the composition Theorem in the heat calculus [36, Proposition 2.6 p. 8] and hence this means for every $p\in M$, there are local coordinates $U\ni p$ s.t. :

$$K^{\star(k+1)}(t,x,y)=t^{-\frac{d+2}{2}+(k+1)}\tilde{A}(t,\frac{x-y}{\sqrt{t}},y)$$

where $\tilde{A}\in C^\infty\left([0,\infty)_{\frac{1}{2}}\times\mathbb{R}^d\times U,E\boxtimes E^*\right)$ by definition of the elements in the heat calculus $\Psi_H^\bullet(M,E)$. Therefore by definition of $\star$, we find :

$$S(s;V_1,\dots,V_{k+1})=\int_0^1 t^s\int_{t\Delta_k}Tr\left(e^{-(t-t_k)\Delta}V\dots Ve^{-t_1\Delta}V\right)dt=\int_0^1 t^s\left(\int_M\left(K^{\star k+1}\right)(t,x,x)dv\right)dt$$

where $\int_0^1 t^s\left(K^{\star k+1}\right)(t,x,x)dt=\int_0^1 t^{s-\frac{d+2}{2}+(k+1)}\tilde{A}(t,0,x)dt$ in local coordinates on M and the r.h.s is Riemann integrable in t near $s=0$. Hence by Fubini, the term S is holomorphic near $s=0$ and given by absolutely convergent integrals.

In the fermionic case, we start from the fact that in local coordinates, $e^{-t\Delta}(x,y)=t^{-\frac{d}{2}}\tilde{A}(t,\frac{x-y}{\sqrt{t}},y)$ where $\tilde{A}\in C^\infty\left([0,\infty)_{\frac{1}{2}}\times\mathbb{R}^d\times U,E_+\boxtimes E_+^*\right)$. From the observation that

$$D_{y^i}t^{-\frac{d}{2}}\tilde{A}(t,\frac{x-y}{\sqrt{t}},y)=t^{-\frac{d+1}{2}}(y^i-x^i)\left(D_{X^i}\tilde{A}\right)(t,\frac{x-y}{\sqrt{t}},y)+t^{-\frac{d}{2}}\left(D_{y^i}\tilde{A}\right)(t,\frac{x-y}{\sqrt{t}},y),$$

we deduce that $K=e^{-t\Delta}D^*A\in\Psi_H^{-\frac{1}{2}}(M,E_+)$. Then for $k+1>d$, by composition in the heat calculus, we find that :

$$\begin{aligned}S(s;D^*A_1,\dots,D^*A_{k+1}) &= \int_0^1 t^s\int_{t\Delta_k}Tr\left(e^{-(t-t_k)\Delta}D^*A\dots e^{-t_1\Delta}D^*A\right)dt\\ &= \int_0^1 t^s\left(\int_M\left(K^{\star k+1}\right)(t,x,x)dv\right)dt\end{aligned}$$

where $\int_0^1 t^s \left(K^{\star k+1}\right)(t,x,x)dt = \int_0^1 t^{s-\frac{d+2}{2}+\frac{k+1}{2}}\tilde{A}(t,0,x)dt$ in local coordinates and the r.h.s is Riemann integrable in t near $s=0$. Hence by Fubini, the term S is holomorphic near $s=0$ and given by absolutely convergent integrals.

In both cases, we find that $\lim_{s\to 0} I\left(s;V_1,\dots,V_{k+1}\right) = \int_{[0,\infty)^{k+1}} Tr\left(e^{-u_{k+1}\Delta}V_1\dots e^{-u_1\Delta}V_{k+1}\right)\prod_{e=1}^{k+1}du_e$ where the right hand side is absolutely convergent.

4.3.4. *Functional derivatives with disjoint supports.* Assume $(V_1,\dots,V_{k+1})$ are such that $\operatorname{supp}(V_1)\cap\dots\cap\operatorname{supp}(V_{k+1})=\emptyset$. Observe that the function $p_t:\xi\in\mathbb{R}\mapsto e^{-t|\xi|^2}$ defines a family $(p_t)_{t\in[0,+\infty)}$ of symbols in $S^0_{1,0}(\mathbb{R})$ such that $p_t \underset{t\to 0}{\to} 1$ in $S^0_{1,0}(\mathbb{R})$ where $p\in S^0_{1,0}(\mathbb{R})$ iff $|\partial_\xi^j p(\xi)|\leqslant C_j(1+|\xi|)^{-j}$ [74, Lemm 1.2 p. 295]. By a result of Strichartz [74, Thm 1.3 p. 296], $p_t(\sqrt{\Delta}) = e^{-t\Delta} \underset{t\to 0^+}{\to} Id$ in $\Psi^0_{1,0}(M,E)$ which implies that the family $(e^{-t\Delta}V_i)_{t\in[0,1]}$ defines a **bounded family of pseudodifferential operators** in $\Psi^0_{1,0}(M,E)$ whose wave front set is uniformly controlled in $T^*_{\operatorname{supp}(V_i)}M$ in the sense that for every pair of cut–off functions $(\chi_1,\chi_2)\in C^\infty(M)^2$, the family $\left(\chi_2 e^{-t\Delta}V_i\chi_1\right)_{t\in[0,+\infty)}$ is bounded in $\Psi^{-\infty}(M,E)$ [13] if $\operatorname{supp}(V_i)\cap\operatorname{supp}(\chi_1)\cap\operatorname{supp}(\chi_2)=\emptyset$, otherwise $(\chi_2 e^{-t\Delta}V_i\chi_1)_{t\in[0,+\infty)}$ is bounded in $\Psi^0_{1,0}(M,E)$. This implies that the family $e^{-(t-t_k)\Delta}V_{k+1}\dots V_2 e^{-t_1\Delta}V_1$, for $\{0\leqslant t_1\leqslant\dots t_k\leqslant t\leqslant 1\}$ is bounded in $\Psi^{-\infty}(M,E)$ by the condition on the support of $(V_i)_{i=1}^{k+1}$. Finally, $\int_{t\Delta_k} Tr\left(e^{-(t-t_k)\Delta}V_{k+1}\dots V_2 e^{-t_1\Delta}V_1\right)=\mathcal{O}(1)$ and

$$\lim_{s\to 0} I(s,V_1,\dots,V_{k+1}) = Tr_{L^2}\left(\Delta^{-1}V_1\dots\Delta^{-1}V_{k+1}\right)$$

where the L^2-trace on the r.h.s is well–defined since $\Delta^{-1}V_1\dots\Delta^{-1}V_{k+1}\in\Psi^{-\infty}(M,E)$. Hence, for every $(k+1)$-uple of open subsets $(U_1,\dots,U_{k+1})$ s.t. $\overline{U_1}\cap\dots\cap\overline{U_{k+1}}=\emptyset$, the multilinear map $(V_1,\dots,V_{k+1})\in C^\infty_c(U_1,End(E))\times\dots\times C^\infty_c(U_{k+1},End(E))\mapsto \lim_{s\to 0}\frac{I(s;V_1,\dots,V_{k+1})}{\Gamma(s+1)} = Tr\left(\Delta^{-1}V_1\dots\Delta^{-1}V_{k+1}\right)$ is multilinear continuous and

$$(V_1,\dots,V_{k+1})\in C^\infty(End(E))^{k+1}\mapsto FP|_{s=0}\frac{I(s;V_1,\dots,V_{k+1})}{\Gamma(s+1)}$$

coincides with the functional derivative $\frac{(-1)^k}{k!}\delta_{V_1}\dots\delta_{V_{k+1}}\log\det_\zeta(\Delta+V)$ of the analytic function $\log\det_\zeta$ on $C^\infty(End(E))$. Observe that $M^{k+1}\setminus d_{k+1}$ is covered by open subsets of the form $U_1\times\dots\times U_{k+1}$ s.t. $\overline{U_1}\cap\dots\cap\overline{U_{k+1}}=\emptyset$. By the multilinear Schwartz kernel 3.3, the above multilinear map is represented by a distribution $\mathcal{R}t_{k+1}\in\mathcal{D}'(M^{k+1},End(E)^{\boxtimes k+1})$ which **coincides** with the product $t_{k+1}=\mathbf{G}(x_1,x_2)\dots\mathbf{G}(x_{k+1},x_1)\in\mathcal{D}'(M^{k+1}\setminus d_{k+1},End(E)^{\boxtimes k+1})$ since $Tr\left(\Delta^{-1}V_1\dots\Delta^{-1}V_{k+1}\right)=\langle t_{k+1},V_1\boxtimes\dots\boxtimes V_{k+1}\rangle$ for $\operatorname{supp}(V_1)\cap\dots\cap\operatorname{supp}(V_{k+1})=\emptyset$ and $\langle .,.\rangle$ is a distributional pairing. In the fermionic case, the discussion is almost identical.

From the above observation, we deduce the following claim which holds true in **both bosonic and fermionic settings** which summarizes all above results :

Proposition 4.7. *Following the notations from definitions (2.4) and (2.3). Let M be a smooth, closed, compact Riemannian manifold of dimension d, and Δ (resp $\Delta=D^*D$) some*

[13] It means the corresponding family of Schwartz kernels are bounded in $C^\infty(M\times M,E\boxtimes E^*)$ for the usual Fréchet topology

generalized Laplacian acting on E (resp E_+) s.t. $\sigma(\Delta) \subset \{Re(z) \geqslant \delta > 0\}$ for bosons (resp fermions).

In the bosonic case, for every invertible $\Delta + V \in \mathcal{A} = \Delta + C^\infty(End(E))$, for every $(V_1, \dots, V_n) \in C^\infty(End(E))^n$, if $n > \frac{d}{2}$ or $supp(V_1) \cap \dots \cap supp(V_n) = \emptyset$ then :

$$\delta_{V_1} \dots \delta_{V_n} \log det_\zeta (\Delta + V) = (-1)^{n-1}(n-1)! Tr_{L^2} \left((\Delta + V)^{-1} V_1 \dots (\Delta + V)^{-1} V_n \right). \quad (4.10)$$

For general $(V_1, \dots, V_n) \in C^\infty(End(E))^n$:

$$\frac{(-1)^{n-1}}{n-1!} \delta_{V_1} \dots \delta_{V_n} \log det_\zeta (\Delta + V) = \langle \mathcal{R} t_n, V_1 \boxtimes \dots \boxtimes V_n \rangle \quad (4.11)$$

*where $\mathcal{R}t_n$ is a **distributional extension** of $t_n = \mathbf{G}(x_1, x_2) \dots \mathbf{G}(x_n, x_1) \in \mathcal{D}'(M^n \setminus d_n, End(E)^{\boxtimes n})$ where $\mathbf{G} \in \mathcal{D}'(M \times M, E \boxtimes E^*)$ is the Schwartz kernel of $(\Delta + V)^{-1}$.*

In the fermionic case, for every invertible $D + A : C^\infty(E_+) \mapsto C^\infty(E_-)$, for every $(A_1, \dots, A_n) \in C^\infty(Hom(E_+, E_-))^n$, if $n > d$ or $supp(A_1) \cap \dots \cap supp(A_n) = \emptyset$ then :

$$\delta_{A_1} \dots \delta_{A_n} \log det_\zeta (\Delta + D^* A) = (-1)^{n-1}(n-1)! Tr_{L^2} \left((D + A)^{-1} A_1 \dots (D + A)^{-1} A_n \right). \quad (4.12)$$

For general $(A_1, \dots, A_n) \in C^\infty(Hom(E_+, E_-))^n$

$$\frac{(-1)^{n-1}}{n-1!} \delta_{A_1} \dots \delta_{A_n} \log det_\zeta (\Delta + D^* A) = \langle \mathcal{R} t_n, A_1 \boxtimes \dots \boxtimes A_n \rangle \quad (4.13)$$

*where $\mathcal{R}t_n$ is a **distributional extension** of $t_n = \mathbf{G}(x_1, x_2) \dots \mathbf{G}(x_n, x_1) \in \mathcal{D}'(M^n \setminus d_n, Hom(E_-, E_+)^{\boxtimes n})$ where $\mathbf{G} \in \mathcal{D}'(M \times M, E_+ \boxtimes E_-^*)$ is the Schwartz kernel of $(D + A)^{-1}$.*

Proof. In the bosonic case, we proved the claim for all $\Delta + V$ s.t. $\sigma(\Delta + V) \subset \{Re(z) \geqslant \delta > 0\}$ since we need the exponential decay of the heat semi–group $e^{-t(\Delta+V)}$ to make the regular part R from definition 4.5 convergent. However, by **analyticity of both sides** of the identity $\delta_{V_1} \dots \delta_{V_n} \log \det_\zeta (\Delta + V) = (-1)^{n-1}(n-1)! Tr_{L^2} \left((\Delta + V)^{-1} V_1 \dots (\Delta + V)^{-1} V_n \right)$ in $V \in C^\infty(M, End(E))$, the claim holds true everywhere on $C^\infty(M, End(E))$ by analytic continuation using the fact that the subset of invertible elements in $\mathcal{A}$ is **connected** [14]. The discussion is identical for the fermionic case. □

4.3.5. *Traces and integrals on configuration space.* In the bosonic case, for $(u_1, \dots, u_{k+1}) \in (0, 1]^{k+1}$, we reformulate the trace term $Tr\left(e^{-u_1 \Delta} V_1 \dots e^{-u_{k+1} \Delta} V_{k+1} \right)$ as an integral over configuration space

$$\int_{M^{k+1}} \left\langle e^{-u_1 \Delta}(x_1, x_2) \dots e^{-u_{k+1} \Delta}(x_{k+1}, x_1), \chi(x_1, \dots, x_{k+1}) \right\rangle dv_{k+1}$$

where $dv_{k+1} \in |\Lambda^{top}| M^{k+1}$, the product $e^{-u_1 \Delta}(x_1, x_2) \dots e^{-u_{k+1} \Delta}(x_{k+1}, x_1)$ on the l.h.s is an element in $C^\infty(M^{k+1}, End(E^*)^{\boxtimes k+1})$, $\chi = V \boxtimes \dots \boxtimes V \in C^\infty(M^{k+1}, End(E)^{\boxtimes k+1})$ is the test object and the $\langle ., . \rangle$ denotes the natural fiberwise pairing between elements of $End(E)^{\boxtimes k+1}$

[14] $\Delta + V$ invertible iff $I + \Delta^{-1} V$ invertible which is true for V in a small neighborhood of $V = 0$. Then consider complex rays $z \in \mathbb{C} \mapsto I + z \Delta^{-1} V$ which are non invertible at isolated values of z since $\Delta^{-1} V$ compact

and $End(E^*)^{\boxtimes k+1}$. Starting from now on in the bosonic case, the test function part χ will be chosen arbitrarily in $C^\infty(M^{k+1}, End(E)^{\boxtimes k+1})$.

In the fermionic case, we will consider the operator $e^{-t\Delta}D^* : C^\infty(E_-) \mapsto C^\infty(E_+)$ which has smoothing kernel when $t > 0$ (since $\Psi^{-\infty}$ is an ideal) hence the trace term $Tr\left(e^{-u_1\Delta}D^*A_1 \dots e^{-u_{k+1}\Delta}D^*A_{k+1}\right)$ is reformulated as the integral over configuration space :

$$\int_{M^{k+1}} \left\langle e^{-u_1\Delta}D^*(x_1,x_2)\dots e^{-u_{k+1}\Delta}D^*(x_{k+1},x_1), \chi(x_1,\dots,x_{k+1})\right\rangle dv_{k+1} \tag{4.14}$$

where $dv_{k+1} \in |\Lambda^{top}|M^{k+1}$, the product $e^{-u_1\Delta}D^*(x_1,x_2)\dots e^{-u_{k+1}\Delta}D^*(x_{k+1},x_1)$ on the l.h.s is an element in $C^\infty(M^{k+1}, Hom(E_-,E_+)^{\boxtimes k+1})$, $\chi = A\boxtimes\cdots\boxtimes A \in C^\infty(M^{k+1}, Hom(E_+,E_-)^{\boxtimes k+1})$ and $\langle .,.\rangle$ denotes the natural fiberwise pairing between elements of $Hom(E_+,E_-)^{\boxtimes k+1}$ and $Hom(E_-,E_+)^{\boxtimes k+1}$.

In what follows, we will localize the study in some open subset of the form U^{k+1} near an element of the form $(x,\dots,x) \in d_{k+1} \subset M^{k+1}$ where $U \subset M, x \in U$ is an open chart that we choose to identify with some bounded open subset U of $\mathbb{R}^d$ making some abuse of notations. Recall that a consequence of the heat calculus is that in local coordinates $e^{-t\Delta}(x,y) = t^{-\frac{d}{2}}\tilde{A}(t, \frac{x-y}{\sqrt{t}}, y), \tilde{A} \in C^\infty([0,+\infty)_{\frac{1}{2}} \times \mathbb{R}^d \times U, E \boxtimes E^*)$ for bosons and $e^{-t\Delta}D^*(x,y) = t^{-\frac{d+1}{2}}\tilde{A}(t, \frac{x-y}{\sqrt{t}}, y), \tilde{A} \in C^\infty([0,+\infty)_{\frac{1}{2}} \times \mathbb{R}^d \times U, E_+ \boxtimes E_-^*)$ for fermions. From this observation on the asymptotics of the kernel $e^{-t\Delta}D^*$ the proofs in both bosonic and fermionic cases are uniform. The only changes occur in the numerology since there is a loss of $t^{-\frac{1}{2}}$ in powers of t in the expansion of $e^{-t\Delta}D^*$.

Definition 4.8. *We define for* $(u,x) = ((u_e)_{e=1}^{k+1}, (x_i)_{i=1}^{k+1}) \in (0,1]^{k+1} \times U^{k+1}$ *:*

$$J(u,x;\chi) = \left\langle \prod_{1\leqslant e\leqslant k+1} \tilde{A}(u_e, \frac{x_{i(e)}-x_{j(e)}}{\sqrt{u_e}}, x_{j(e)}), \chi \right\rangle$$

where $i(e) = e, j(e) = e+1$ *when* $e \in \{1,\dots,k\}$ *and* $i(k+1) = k+1, j(k+1) = 1$*, the bracket* $\langle .,.\rangle$ *denotes the appropriate fiberwise pairing defined above,* $J(.,.,\chi) \in C^\infty\left((0,1]^{k+1} \times U^{k+1}\right)$ *and* J *depends linearly on* χ*.*

Then we can express S from definition 4.5 in terms of J :

$$S(s;\chi) = \int_{\mathbf{\Delta}_{k+1}} (\sum_{e=1}^{k+1} u_e)^s \left(\int_{M^{k+1}} J(u,x;\chi)\right) d^{k+1}u, \chi = V_1 \boxtimes \cdots \boxtimes V_{k+1}.$$

We will next prove that S is a distribution valued in meromorphic functions of the variable s [15, def 3.5 p. 11] based on the methods of [15] using blow–ups.

4.3.6. *Resolving products of heat kernels.* The problem in the definition of S is that the product $\prod_{1\leqslant e\leqslant k+1} \tilde{A}(u_e, \frac{x_{i(e)}-x_{j(e)}}{\sqrt{u_e}}, x_{j(e)})$ is not smooth on $\mathbf{\Delta}_{k+1} \times U^{k+1}$. We choose the test object χ supported in $U^{k+1} \subset M^{k+1}$. We integrate w.r.t the Riemannian volume $dv = \rho d^dx$ with smooth density ρ w.r.t. the Lebesgue measure d^dx hence without loss of generality, we choose to absorb ρ in the test object χ.

Definition 4.9 (Blow–up)**.** *Consider the following change of variables :*

$$\begin{aligned}\beta:(x,h_1,\dots,h_k,t_1,\dots,t_{k+1}) &\longmapsto ((x_1=x,x_i=x+\sum_{j=1}^{i-1}(t_j\dots t_{k+1})h_j)_{i=2}^{k+1},(u_l=\prod_{j=l}^{k+1}t_j^2)_{l=1}^{k+1})\\ U\times\mathbb{R}^{kd}\times[0,1]^{k+1} &\longmapsto U^{k+1}\times\mathbf{\Delta}_{k+1}\end{aligned}$$

[15, def 5.3] *which resolves the singular product in J. We use the short notation* $(x,h,t)=(x,(h_1,\dots,h_k),(t_1,\dots,t_{k+1}))\in U\times\mathbb{R}^{kd}\times[0,1]^{k+1}$.

Replacing in the integral expression of S yields,

$$S(s;\chi)=\int_{[0,1]^{k+1}\times U\times\mathbb{R}^{dk}}((t_1\dots t_k)^2+\dots+1)^s\beta^*J(t,x,h;\chi)t_1\dots t_{k+1}^{2k+2s+1}d^{k+1}td^dxd^{kd}h,$$

where the factor t_{k+1}^{2k+1} comes from $\beta^*\left(d^{k+1}u\right)=2^{k+1}t_1\dots t_{k+1}^{2k+1}d^{k+1}t$. One of the key results from [15, Thm 5.2] is that for every $e\in\{1,\dots,k\}$, the pull–back $\beta^*\left(\frac{x_{i(e)}-x_{j(e)}}{\sqrt{u_e}}\right)$ by the blow–down map β hence β^*J is a **smooth function** on the resolved space $U\times\mathbb{R}^{dk}$. In the bosonic (resp fermionic) case, the change of variables in $\beta^*\left(\prod_{e=1}^{k+1}u_e^{-\frac{d}{2}}\right)$ (resp $\beta^*\left(\prod_{e=1}^{k+1}u_e^{-\frac{d+1}{2}}\right)$) brings a factor of the form $\prod_{1\leqslant l\leqslant k+1}(t_l\dots t_{k+1})^{-d}$ (resp $\prod_{1\leqslant l\leqslant k+1}(t_l\dots t_{k+1})^{-d-1}$) and the Jacobian of the variable change yields a factor $(t_1\dots t_{k+1})^d\dots(t_kt_{k+1})^d$ from which we can extract the power of t_{k+1} to be equal to t_{k+1}^{-d} (resp t_{k+1}^{-d-k-1}). Replacing in the integral formula yields the following identity [15, Proposition 5.2],

$$\begin{aligned}S(s;\chi)=&\int_{[0,1]^{k+1}}((t_1\dots t_k)^2+\dots+1)^sP(t_1,\dots,t_k)\\ &\times\left(\int_{\mathbb{R}^{d(k+1)}}A(t,x,h;\chi)d^dxd^{kd}h\right)t_{k+1}^{2s+2k+1-d}dt_1\dots dt_{k+1}\end{aligned}\tag{4.15}$$

where P is a polynomial function whose explicit expression is irrelevant and $A(.;\chi)=\beta^*J(.;\chi)\in C^\infty([0,1]^{k+1}\times U\times\mathbb{R}^{dk})$, for fermions we get t_{k+1}^{2s+k-d} in factor under the integral. As in [15, Lemma 5.4], A depends **linearly** on χ by the explicit formula $\pi^*\chi(t,x,h)=\chi(x,x+\sum_{j=1}^{i-1}(t_j\dots t_{k+1})h_j)_{i=2}^{k+1}$. We find

Lemma 4.10. *Under the previous notations, in both bosonic and fermionic case, the quantity $\partial^p_{t_{k+1}}A(t,x,h)|_{t_{k+1}=0}$ depends linearly on p-jets of the coefficients of χ along the diagonal $d_{k+1}\subset M^{k+1}$.*

In the following paragraph, we shall prove that both $\chi\mapsto S(s;\chi)$ and $\chi\mapsto I(s;\chi)$ are distributions valued in meromorphic germs at $s=0$.

4.3.7. *The bad case when $1\leqslant k\leqslant\frac{d}{2}$ (resp $1\leqslant k\leqslant d$) for bosons (resp fermions) case and integration by parts.* In the bosonic case, if $k\leqslant\frac{d}{2}$, the factor $t_{k+1}^{2(s+k)+1-d}$ appearing in factor of A is potentially divergent since near $s=0$ the exponent $2(s+k)+1-d\geqslant 2k+1-d$ is no longer necessary >-1. Then as usual in Riesz regularization, we need to Taylor expand A w.r.t the variable t_{k+1} up to order p in such a way that $(p+1)+2k+1-d > -1 \implies p+1 > d-2(k+1)$. This yields

$$\begin{aligned}
& \int_{[0,1]^{k+1}} ((t_1\dots t_k)^2+\dots+1)^s P(t_1,\dots,t_k) \\
\times & \left(\int_{\mathbb{R}^{d(k+1)}} A(t,x,h;\chi) d^dx d^{kd}h\right) t_{k+1}^{2(s+k)+1-d} dt_1\dots dt_{k+1} \\
= & \sum_{p=0}^{\sup(d-2(k+1),0)} \int_{[0,1]^k} ((t_1\dots t_k)^2+\dots+1)^s P(t_1,\dots,t_k) \frac{\int_{\mathbb{R}^{d(k+1)}} \partial^p_{t_{k+1}} A(t,x,h;\chi)|_{t_{k+1}=0} d^dx d^{kd}h}{s+k+p+1-\frac{d}{2}} dt_1\dots dt_k \\
+ & \text{ holomorphic at } s=0
\end{aligned}$$

where the holomorphic part depends linearly on the $(d-3)$-jet of χ.

This implies that in the general case $FP|_{s=0}\frac{1}{\Gamma(s+1)}I(s,\chi)$ depends linearly on the $(d-3)$-jets of χ and when $\text{supp}(\chi)$ does not meet the deepest diagonal $d_{k+1}\subset M^{k+1}$ we already know that $FP|_{s=0}\frac{1}{\Gamma(s+1)}I(s,\chi) = \int_{M^{k+1}}\left\langle \prod_{e=1}^{k+1}\mathbf{G}(x_{i(e)},x_{j(e)}),\chi\right\rangle \prod_{i=1}^{k+1} dv(x_i)$. Altogether, this proves that $\frac{(-1)^k}{k!}\delta^{k+1}\log\det_\zeta(\Delta)$ is a **distributional extension** of $\mathbf{G}(x_1,x_2)\dots\mathbf{G}(x_{k+1},x_1)$ of **order at most** $(d-3)$. The fermionic case is similar only the numerology differs, we need to expand coefficients of χ at order p in t_{k+1} so that $p+1>d-2$ therefore $FP|_{s=0}\frac{1}{\Gamma(s+1)}I(s,\chi)$ depends linearly on $(d-1)$-jets of the coefficients of χ.

4.3.8. *Bounds on the Fourier transform of the singular term S.* In this part the bosonic and fermionic cases are similar and therefore we restrict to the former case for simplicity. Using the above notations, we should study $S(s;\chi)$ where the test function part χ is of the form $\chi=\psi(x_1)e^{ix_1\xi_1}\dots\psi(x_{k+1})e^{ix_{k+1}\xi_{k+1}}$ where ψ has small support in the coordinate chart U of M and for large $(\xi_1,\dots,\xi_{k+1})$ in some closed conic set V. After the change of variables of definition 4.9, the exponential factor becomes

$$\exp\left(ix(\xi_1+\dots+\xi_{k+1})+i\sum_{e=2}^{k+1}\sum_{j=1}^{e-1}(t_j\dots t_{k+1})h_j\xi_e\right)$$
$$=\exp\left(ix(\xi_1+\dots+\xi_{k+1})+i\sum_{j=1}^{k}(t_j\dots t_{k+1})h_j(\sum_{j+1\leqslant e\leqslant k+1}\xi_e)\right)$$

and the term $A(t,x,h;\chi)=A(t,x,h;\psi^{\boxtimes k+1})e^{\left(ix\sum_{j=1}^{k+1}\xi_j+i\sum_{e=2}^{k+1}\sum_{j=1}^{e-1}(t_j\dots t_{k+1})h_j\xi_e\right)}$ so the interesting term in factor of S that we should integrate by parts w.r.t. t_{k+1} reads

$$\begin{aligned}
& |\partial^p_{t_{k+1}}\int_{\mathbb{R}^{d(k+1)}} e^{\left(ix\sum_{j=1}^{k+1}\xi_j+i\sum_{e=2}^{k+1}\sum_{j=1}^{e-1}(t_j\dots t_{k+1})h_j\xi_e\right)} A(t,x,h;\psi^{\boxtimes k+1}) d^dx d^{kd}h| \\
\leqslant & C(1+K\sum_{e=1}^{k+1}|\xi_e|)^p \sup_{1\leqslant j\leqslant p}\sup_{t\in[0,1]^{k+1}} |\widehat{\partial^j_t A}(t,\sum_{e=1}^{k+1}\xi_e,(t_j\dots t_{k+1})\sum_{e=j+1}^{k+1}\xi_e;\psi^{\boxtimes k+1})| \leqslant C_N(1+|\sum_{e=1}^{k+1}\xi_e|)^{-N}
\end{aligned}$$

uniformly in $t\in[0,1]^{k+1}$ for all N by **smoothness of A and its derivatives $\partial^j_t A$ in the x variable**. Assume that $(\xi_1,\dots,\xi_{k+1})$ belongs to some cone V which does not meet the hyperplane $\{\sum_{i=1}^{k+1}\xi_i=0\}$, then there exists a constant C such that for every $(\xi_1,\dots,\xi_{k+1})\in V$ satisfying $\sum_{i=1}^{k+1}|\xi_i|^2\geqslant R$: $|\sum_{i=1}^{k+1}\xi_i|\geqslant\varepsilon\left(1+\sum_{i=1}^{k+1}|\xi_i|\right)$. Therefore we obtain the estimate :

$$|\partial_{t_{k+1}}^p\int_{\mathbb{R}^{d(k+1)}}e^{\left(ix\sum_{j=1}^{k+1}\xi_j+i\sum_{e=2}^{k+1}\sum_{j=1}^{e-1}(t_j\dots t_{k+1})h_j\xi_e\right)}A(t,x,h;\psi^{\boxtimes k+1})d^dxd^{kd}h|\leqslant C_N\varepsilon^{-N}(1+\sum_{e=1}^{k+1}|\xi_e|)^{-N}$$

uniformly in $t\in[0,1]^{k+1}$. Finally, this means that for any $(x,\dots,x;\xi_1,\dots,\xi_{k+1})\in WF\left(FP|_{s=0}S(s,.)\right)$, we must have $\xi_1+\dots+\xi_{k+1}=0$ which implies the wave front set bound

$$\left(WF\left(FP|_{s=0}S(s,.)\right)\cap T^\bullet_{d_{k+1}}M^{k+1}\right)\subset N^*\left(d_{k+1}\subset M^{k+1}\right)$$

over the diagonal d_{k+1}. In the next paragraph, we will use these bounds on the Fourier transform of S to estimate the wave front set of the Schwartz kernel of the functional derivatives over the diagonal.

4.3.9. *Wave front bounds.* The next step is to use the above methods to bound the wave front set of the Schwartz kernels of the functional derivatives. An important application of the blow–up techniques is to estimate the wave front set of the extensions $\mathcal{R}t_n\in\mathcal{D}'(M^n)$ over the deep diagonal $d_n\subset M^n$ which is proved to be contained in the conormal bundle $N^*\left(d_n\subset M^n\right)$.

Proposition 4.11. *Following the notations from definitions (2.4) and (2.3). Let M be a smooth, closed, compact Riemannian manifold of dimension d, $E\mapsto M$ some Hermitian bundle over M.*

For every invertible generalized Laplacian $\Delta+V\in\mathcal{A}=\Delta+C^\infty(End(E))$ acting on E s.t. $\sigma(\Delta)\subset\{Re(z)\geqslant\delta>0\}$, set $\mathbf{G}\in\mathcal{D}'(M\times M,E\boxtimes E^*)$ to be the Schwartz kernel of $(\Delta+V)^{-1}$. The Schwartz kernel of the functional derivative of $\log\det_\zeta$ defined as $\mathcal{R}t_n=\frac{(-1)^{n-1}}{(n-1)!}\delta^n\log\det_\zeta(\Delta+V)$ is a distributional extension of the product $t_n=\prod_{e\in E(G)}\mathbf{G}(x_{i(e)},x_{j(e)})\in\mathcal{D}'(M^n\setminus d_n,End(E^*)^{\boxtimes n})$ and satisfies the wave front bound

$$\boxed{\left(WF\left(\mathcal{R}t_n\right)\cap T^\bullet_{d_n}M^n\right)\subset N^*\left(d_n\subset M^n\right).}\tag{4.16}$$

Proof. We need to show that the distribution $\mathcal{R}t_n$ defined as $\frac{(-1)^n}{(n-1)!}\frac{d}{ds}|_{s=0}\delta_{V_1}\dots\delta_{V_n}Tr\left((\Delta+V)^{-s}\right)=\langle\mathcal{R}t_n,V_1\boxtimes\dots\boxtimes V_n\rangle$ satisfies the wave front bound $WF(\mathcal{R}t_n)\cap T^\bullet_{d_n}M^n\subset N^*\left(d_n\subset M^n\right)$. We start from the expression

$$\frac{1}{\Gamma(s+1)}\int_{[0,\infty)^n}(u_1+\dots+u_n)^s\int_{M^n}\left\langle e^{-u_1\Delta}\dots e^{-u_n\Delta},\Psi\right\rangle\prod_{e=1}^n du_e.$$

We work on a local chart U^{k+1} where we choose the test section χ to be equal to $\chi=\psi(x_1)e^{ix_1\xi_1}\dots\psi(x_n)e^{ix_n\xi_n}$ where $\psi\in C^\infty_c(U)$ is supported on some chart U. There is a competition between :

(1) integration of heat kernels on $[1,+\infty)$ which yields smoothing operators in the sense the family $\left(e^{-u\Delta}(x,y)\right)_{u\in[1,+\infty)}$ is bounded in $C^\infty(M\times M,E\boxtimes E^*)$ since $e^{-u\Delta}=$

$e^{-\frac{1}{4}\Delta}\underbrace{e^{-(u-\frac{1}{2})\Delta}}_{\text{bounded}}e^{-\frac{1}{4}\Delta}$ where the term in the middle is uniformly bounded in $\mathcal{B}(L^2,L^2)$ by spectral assumption and both factors $e^{-\frac{1}{4}\Delta}$ on the left and right are smoothing operators in (x,y) variable,

(2) integration on $[0,1]$ which yields singular distributions whose wave front set is conormal in the sense that the family $\left(e^{-u\Delta}(x,y)\right)_{u\in(0,1]}$ is a bounded family of distributions in $\mathcal{D}'_{N^*(d_2\subset M^2)}(M\times M, E\boxtimes E^*)$ [15] which is the space of distributions whose wave front set is contained in the conormal bundle $N^*\left(d_2\subset M^2\right)$.

Introduce a first decomposition where we sum over permutations S_n of $\{1,\dots,n\}$ in the second sum :

$$\begin{aligned} &\int_{[0,\infty)^n}(u_1+\dots+u_n)^s\int_{M^n}\left\langle e^{-u_1\Delta}\dots e^{-u_n\Delta},\chi\right\rangle\prod_{e=1}^n du_e \\ = \;&\sum_{k=0}^n\frac{1}{n!}\sum_{\sigma\in S_n}\int_{[0,1]^k\times[1,+\infty)^{n-k}}(u_1+\dots+u_n)^s\int_{M^n}\left\langle e^{-u_{\sigma(1)}\Delta}\dots e^{-u_{\sigma(n)}\Delta},\chi\right\rangle\prod_{e=1}^n du_e\end{aligned}$$

Without loss of generality, we only treat the terms corresponding to the identity permutation of S_n. When $k<n$, using the hypocontinuity of the product of distributions whose wave front set is fixed [14, Thm 6.1 p. 219] and the fact that the family $e^{-u_i\Delta}(x_i,x_{i+1})$, viewed as distribution on M^n, is bounded in $\mathcal{D}'_{N^*\left(d_{\{i,i+1\}}\subset M^n\right)}(M^n)$ where $N^*\left(d_{\{i,i+1\}}\subset M^n\right)$ is the conormal of the diagonal $d_{\{i,i+1\}}=\{x_i=x_{i+1}\}\subset M^n$, we note that the distributional product $\left(e^{-u_1\Delta}\dots e^{-u_k\Delta}\right)_{(u_1,\dots,u_k)\in[0,1]^k}$ is bounded in $\mathcal{D}'_\Gamma(M^n)$ for $\Gamma=\bigcup_I N^*\left(d_I\subset M^n\right)\subset T^\bullet M^n$, where the union runs over the sets $I=\{i,\dots,j\}$, where $\{i,\dots,j\}$ contains the arithmetic progression from i to j, for $1\leqslant i<j\leqslant k$. Then it follows immediately that for $k<n$,

$$WF\left(\int_{[0,1]^k\times[1,+\infty)^{n-k}}(u_1+\dots+u_n)^s\int_{M^n}\left\langle e^{-u_1\Delta}\dots e^{-u_n\Delta},\chi\right\rangle\prod_{e=1}^n du_e\right)\cap T^*_{d_n}M^n\subset N^*\left(d_n\subset M^n\right).$$

For the term where $k=n$, the result follows simply from the bounds on the Fourier transform of the singular term $\left(WF(FP|_{s=0}S(s,.))\cap T^*_{d_n}M^n\right)\subset N^*\left(d_n\subset M^n\right)$ from paragraph 4.3.8. □

4.4. **Factorization formula relating** $\det_\zeta$ **and Gohberg–Krein's determinants** $\det_p$**.** We give the proof for bosons and write the factorization formula for fermions for simplicity since the discussion is almost similar in both cases. Lemma 3.7 implies that for z small enough, the series $\sum_{n\geqslant\frac{d}{2}+1}\frac{(-1)^{n+1}z^n}{n}Tr_{L^2}\left(\left(\Delta^{-1}V\right)^n\right)$ converges and equals $\log\det_p\left(Id+z\Delta^{-1}V\right)$ for $p=[\frac{d}{2}]+1$ where $\det_p$ is Gohberg–Krein's determinant whose properties are recalled in Lemma 3.7 of subsection 3.3. In particular, $\det_p\left(Id+\Delta^{-1}V\right)$ is analytic in $V\in C^\infty(End(E))$ and vanishes iff $\Delta+V$ is non invertible and $z\mapsto\det_p\left(Id+z\Delta^{-1}V\right)$ is an entire function.

[15]in the sense of the seminorms in [14, p. 204]

It follows from Proposition 4.7 that functional derivatives of $\log\det_p(I+\Delta^{-1}V)$ and $\log\det_\zeta(\Delta+V)$ coincide when $k>\frac{d}{2}$ (resp $k>d$) in the bosonic (resp fermionic) case. Hence for bosons and for z small enough, we have the identity

$$\log\det_\zeta(\Delta+zV)=P(zV)+\log\det_p(Id+z\Delta^{-1}V)$$

where both sides are holomorphic germs and $P(zV)$ is a polynomial in z of degree $[\frac{d}{2}]$. Therefore for every fixed V, $z\mapsto\det_\zeta(\Delta+zV)$ extends uniquely as an entire function with same divisor as $\det_p(Id+z\Delta^{-1}V)$. Since $V\mapsto\det_\zeta(\Delta+V)$ is locally bounded near $V=0$ by analyticity of $V\mapsto\det_\zeta(\Delta+V)$, Proposition 3.6 implies that

$$\log\det_\zeta(\Delta+V)=P(V)+\log\det_p(Id+\Delta^{-1}V) \tag{4.17}$$

for V close enough to 0 where P is a **continuous polynomial function** of V. Equation (4.17) together with the fact that $H\mapsto\det_p(I+H)$ is an entire function on the Schatten ideal $\mathcal{I}_p$ vanishing exactly over noninvertible $I+H$, proves that $V\mapsto\det_\zeta(\Delta+V)=e^{P(V)}\det_p(Id+\Delta^{-1}V)$ extends uniquely as a complex analytic function on $\mathcal{A}$ vanishing exactly over non invertible elements. Then by Proposition 4.7, the functional derivatives $\frac{(-1)^{n-1}}{n-1!}\delta^n\log\det_\zeta(\Delta+V)|_{V=0}$ are **distributional extensions** of the distributions $t_n=\mathbf{G}(x_1,x_2)\dots\mathbf{G}(x_n,x_1)$. It follows that $P(V)=\sum_{1\leqslant n\leqslant\frac{d}{2}}\frac{(-1)^{n+1}}{n}\left\langle\mathcal{R}t_n,V^{\boxtimes n}\right\rangle$ where $WF(\mathcal{R}(t_n))$ satisfies the bound $WF(\mathcal{R}(t_n))\cap T^*_{d_n}M^n\subset N^*(d_n\subset M^n)$ by Proposition 4.11. This concludes the proof that $\det_\zeta$ admits the representation 2.17. The proof for fermions is similar and yields the factorization formula $\det_\zeta(\Delta+D^*A)=\exp(P(A))\det_p\left(I+\Delta^{-1}D^*A\right)$ for $p=d+1$ and P a continuous polynomial of degree d on $C^\infty(Hom(E_+,E_-))$.

5. Proof of Theorem 2.

As above, we give the proof for bosons since the fermion case is similar and presents no extra difficulties.

5.0.1. *Any element of the form $\mathcal{R}\det=e^{Polynomial}\det_\zeta$ solves Problem 2.7.* The zeta determinants from definition 2.3 are solutions of problem 2.7 by Theorem 1 and Proposition 4.7 where we found the second functional derivatives of $\det_\zeta$ to be equal to

$$\delta_{V_1}\delta_{V_2}\log\det_\zeta(\Delta+V)=Tr_{L^2}((\Delta+V)^{-1}V_1(\Delta+V)^{-1}V_2)$$

when $(V_1,V_2)\in C^\infty(End(E))^2$ have disjoint supports and $\sigma(\Delta+V)\subset\{Re(z)\geqslant\delta>0\}$. Therefore, for any local polynomial functional $P\in\mathcal{O}_{loc,[\frac{d}{2}]}(J^mEnd(E))$ of degree $[\frac{d}{2}]$, the map $V\mapsto\mathcal{R}\det(\Delta+V)=\exp(P(V))\det_\zeta(\Delta+V)$ satisfies $\delta_{V_1}\delta_{V_2}\log\mathcal{R}\det(\Delta+V)=\delta_{V_1}\delta_{V_2}\log\det_\zeta(\Delta+V)$ where $\delta_{V_1}\delta_{V_2}P(V)=0$ since (V_1,V_2) have disjoint support and P is local [6, Prop V.5 p. 16]. This means

$$\begin{aligned}\delta_{V_1}\delta_{V_2}\log\mathcal{R}\det(\Delta+V) &= \delta_{V_1}\delta_{V_2}\log\det_\zeta(\Delta+V)\\ =Tr_{L^2}((\Delta+V)^{-1}V_1(\Delta+V)^{-1}V_2) &= \int_{M\times M}t_2(x_1,x_2)V(x_1)V(x_2)dv(x_1)dv(x_2)\end{aligned}$$

and the wave front bound from Proposition 4.11 shows that the kernel of $\delta^2\log\mathcal{R}\det(\Delta+V)=\delta^2\log\det_\zeta(\Delta+V)+\delta^2P(V)$ is a distribution $\mathcal{R}t_2$ satisfying $WF(\mathcal{R}t_2)\cap T^\bullet_{d_2}M^2\subset$

$N^*\left(d_2\subset M^2\right)$ where we used the fact that $WF\left(\delta^2P(V)\right)\subset N^*\left(d_2\subset M^2\right)$ by [6, Lemma VI.9 p.19]. For the moment, we found $\mathcal{R}\det$ solves the equations (2.13) and (2.14) and equation (2.12) is easily satisfied by the factorization formula $\mathcal{R}\det(\Delta+V)=\det_\zeta(\Delta+V)\,e^{P(V)}=e^{(P+Q)(V)}\det_{[\frac{d}{2}]+1}\left(Id+\Delta^{-1}V\right),\deg(P+Q)\leqslant[\frac{d}{2}]$ and the properties of $\det_p$. The last step is to use the factorization formula $\det_\zeta(\Delta+V)=e^{Q(V)}\det_{[\frac{d}{2}]+1}\left(Id+\Delta^{-1}V\right)$ from the previous section and the bound

$$|\det_{[\frac{d}{2}]+1}\left(Id+\Delta^{-1}V\right)|\leqslant e^{K_1\|\Delta^{-1}V\|^{[\frac{d}{2}]+1}_{[\frac{d}{2}]+1}}\leqslant e^{K_1\left(\|\Delta^{-1}\|_{[\frac{d}{2}]+1}\|V\|_{C^0}\right)^{[\frac{d}{2}]+1}}$$

which results from [66, b) Thm 9.2 p. 75] for the norm $\|.\|_{[\frac{d}{2}]+1}$ in the Schatten ideal $\mathcal{I}_{[\frac{d}{2}]+1}$ and the fact that $\Delta^{-1}\in\Psi^{-2}(M,E)$ belongs to $\mathcal{I}_{[\frac{d}{2}]+1}$ since $\Delta^{-[\frac{d}{2}]-1}\in\mathcal{I}_1$ [24, Prop B.20] and Hölder's inequality $\|\Delta^{-1}V\|_{[\frac{d}{2}]+1}\leqslant\|\Delta^{-1}\|_{[\frac{d}{2}]+1}\|V\|_{C^0}$. From the above facts, we deduce the bound :

$$|\mathcal{R}\det(\Delta+V)|\leqslant|\det_\zeta(\Delta+V)||e^{P(V)}|\leqslant Ce^{K\|V\|^{[\frac{d}{2}]+1}_{C^m}}$$

for some $C,K>0$ independent of V which proves the bound (2.11). Finally, $\mathcal{R}\det$ solves problem 2.7.

5.0.2. *Any renormalized determinant is of the form $e^{Polynomial}\det_\zeta$.* Let $\mathcal{R}\det$ be any other solution of problem 2.7, then for every V, the entire functions $z\mapsto\mathcal{R}\det(\Delta+zV)$ and $z\mapsto\det_\zeta(\Delta+zV)$ have same divisor. It follows that the ratio $z\mapsto\frac{\mathcal{R}\det(\Delta+zV)}{\det_\zeta(\Delta+zV)}$ is an entire function **without zeros** on $\mathbb{C}$ which satisfies the bound

$$|\frac{\mathcal{R}\det(\Delta+zV)}{\det_\zeta(\Delta+zV)}|\leqslant Ce^{K|z|^{[\frac{d}{2}]+1}\|V\|^{[\frac{d}{2}]+1}_{C^m}},m=d-3.$$

By the uniqueness part of Hadamard's Theorem 4, this implies that for every fixed V, $z\mapsto\frac{\mathcal{R}\det(\Delta+zV)}{\det_\zeta(\Delta+zV)}=e^{P(z;V)}$ where P is a polynomial of degree $[\frac{d}{2}]+1$ in z. We already know the map $V\mapsto\log\mathcal{R}\det(\Delta+V)-\log\det_\zeta(\Delta+V)$ is analytic near $V=0$ hence locally bounded near $V=0$ and also the above shows that for every fixed V, $z\mapsto\log\mathcal{R}\det(\Delta+zV)-\log\det_\zeta(\Delta+zV)$ is a polynomial of degree $[\frac{d}{2}]+1$ in z. By proposition 3.6, this implies that the difference $\log\mathcal{R}\det(\Delta+V)-\log\det_\zeta(\Delta+V)=P(V)$ where P is actually a **continuous polynomial function** in V of degree $[\frac{d}{2}]+1$. But condition 2.12 imposes the derivatives $(\frac{d}{dz})^{[\frac{d}{2}]+1}\log\mathcal{R}\det(\Delta+zV)$ and $(\frac{d}{dz})^{[\frac{d}{2}]+1}\log\det_\zeta(\Delta+zV)$ to coincide at $z=0$ hence P is in fact of order $[\frac{d}{2}]$. It remains to show that P is local. But the fact that both $\log\mathcal{R}\det(\Delta+V)$ and $\log\det_\zeta(\Delta+V)$ are solutions of functional equation 2.13 implies that $\delta_{V_1}\delta_{V_2}P(V)=0$ if $\text{supp}(V_1)\cap\text{supp}(V_2)=\emptyset$. Observe that $V\mapsto\delta^2\log\left(\frac{\mathcal{R}\det(\Delta+V)}{\det_\zeta(\Delta+V)}\right)=\delta^2P(V)\in\mathcal{D}'(M\times M)$ is polynomial in V valued in distributions on $M\times M$ with wave front set in $N^*\left(d_2\subset M^2\right)$. To extract the homogeneous part , we use the finite difference operator Δ_V defined in the proof of 3.6, the element $\frac{\Delta_V^{n-2}\delta^2P}{(n-2)!}=P_n(V,\dots,V,.,.)$ has also wave front set in $N^*\left(d_2\subset M^2\right)$ thus $V\mapsto P_n(V)$ satisfies the assumptions of lemma 9.1 proved in appendix. Therefore $V\mapsto\left\langle\tilde{P}_n,V^{\boxtimes n}\right\rangle$ is a local functional which equals $\int_M\Lambda_n(j^mV(x))dv(x)$ where $\Lambda_n(j^kV(x))$ depends on the m–jets of V at x for $m=d-3$ and is homogeneous of degree n in V. It is important to stress that the function Λ_n is not uniquely defined [16] but the **functional** is uniquely defined. Then locality of P together with the representation formula for $\det_\zeta$ from Theorem 1 implies that any solution of problem 2.7 has the form given by equation 2.17. The infinite product representation is an easy consequence of the representation of Gohberg–Krein's's determinants $\det_p$ as infinite products.

To complete the proof of our Theorem, it remains to show that any $\mathcal{R}\det$ solution of Problem 2.7 is obtained by a renormalization with subtraction of local counterterms in the sense of the third property in 2 which is the goal of the next section.

6. Local renormalization and Theorem 6 on Gaussian Free Field representation.

We follow the notations from subsubsection 2.0.8 where we explained the notion of subtraction of local counterterms. The aim of this section is to show the third claim from Theorem 2 namely that all $\mathcal{R}\det$ solutions from problem 2.7 are obtained from renormalization by subtraction of **local counterterms** which concludes the proof from Theorem 2 : there exists a generalized Laplacian Δ with heat operator $e^{-t\Delta}$ and a family $Q_\varepsilon \in \mathcal{O}_{loc,[\frac{d}{p}]}\left(J^m Hom(E_+,E_-)\right)\otimes_{\mathbb{C}}\mathbb{C}[\varepsilon^{-\frac{1}{2}},\log(\varepsilon)]$ such that :

$$\mathcal{V}\mapsto \mathcal{R}\det\left(P+\mathcal{V}\right)=\lim_{\varepsilon\to 0^+}\exp\left(Q_\varepsilon(\mathcal{V})\right)\det_F\left(I+e^{-2\varepsilon\Delta}P^{-1}\mathcal{V}\right). \tag{6.1}$$

6.1. Extracting singular parts.

In this subsection, we shall use the methods of [15] based on blow–ups to extract the singular parts of regularized traces $Tr_{L^2}\left(\left(e^{-2\varepsilon\Delta}\Delta^{-1}V\right)^n\right)$ to show :

Lemma 6.1. *In the bosonic case, for every $V\in C^\infty(M,End(E))$, we have an asymptotic expansion*

$$Tr_{L^2}\left(\left(e^{-2\varepsilon\Delta}\Delta^{-1}V\right)^n\right)=P_\varepsilon(V)+\mathcal{O}(1)$$

where $P_\varepsilon=\int_M\Lambda_\varepsilon(j^mV)dv\in\mathcal{O}_{loc,[\frac{d}{2}]}\left(J^m End(E)\right)\otimes_{\mathbb{C}}\mathbb{C}[\varepsilon^{-\frac{1}{2}},\log(\varepsilon)]$ and $m=d-3$ and

$$\begin{aligned}(V_1,\dots,V_{k+1})\in C^\infty(End(E))^{k+1} &\mapsto FP|_{\varepsilon=0}Tr_{L^2}\left(e^{-2\varepsilon\Delta}\Delta^{-1}V_1\dots e^{-2\varepsilon\Delta}\Delta^{-1}V_{k+1}\right)\\ &= \lim_{\varepsilon\to0^+}Tr_{L^2}\left(\left(e^{-2\varepsilon\Delta}\Delta^{-1}V\right)^n\right)-P_\varepsilon(V)\end{aligned}$$

is well–defined and multilinear continuous.

For fermions, for every $A\in C^\infty(Hom(E_+,E_-))$, we have an asymptotic expansion

$$Tr_{L^2}\left(\left(e^{-2\varepsilon\Delta}\Delta^{-1}D^*A\right)^n\right)=P_\varepsilon(A)+\mathcal{O}(1)$$

where $P_\varepsilon=\int_M\Lambda_\varepsilon(j^mA)dv\in\mathcal{O}_{loc,d}\left(J^m End(E)\right)\otimes_{\mathbb{C}}\mathbb{C}[\varepsilon^{-\frac{1}{2}},\log(\varepsilon)]$ and $m=d-1$ and

$$\begin{aligned}(A_1,\dots,A_{k+1})\in C^\infty(Hom(E_+,E_-))^{k+1} &\mapsto FP|_{\varepsilon=0}Tr_{L^2}\left(e^{-2\varepsilon\Delta}\Delta^{-1}D^*A_1\dots e^{-2\varepsilon\Delta}\Delta^{-1}D^*A_{k+1}\right)\\ &= \lim_{\varepsilon\to0^+}Tr_{L^2}\left(\left(e^{-2\varepsilon\Delta}\Delta^{-1}D^*A\right)^n\right)-P_\varepsilon(A)\end{aligned}$$

[16] Only up to boundary terms

is well–defined and multilinear continuous.

Proof. We prove the claim only for bosons, the fermionic case is similar. In this lemma, we shall use the following notation, for two functions $a(\varepsilon), b(\varepsilon)$, we shall note $a \simeq b$ if $b-a = \mathcal{O}(1)$ when $\varepsilon \to 0^+$. We start from the identity :

$$Tr_{L^2}\left(e^{-2\varepsilon\Delta}\Delta^{-1}V_1 \dots e^{-2\varepsilon\Delta}\Delta^{-1}V_{k+1}\right) \simeq \int_{[\varepsilon,1]^{k+1}} Tr_{L^2}\left(e^{-u_1\Delta}V_1 \dots e^{-u_{k+1}\Delta}V_{k+1}\right) du_1 \dots du_{k+1}$$

as a direct consequence of $e^{-2\varepsilon\Delta}\Delta^{-1} = \int_{2\varepsilon}^{\infty} e^{-t\Delta}dt$ and since $\int_1^\infty e^{-t\Delta}dt \in \Psi^{-\infty}$. Now without loss of generality and using the symmetry of the integral, we may assume that we work in the Hepp sector $\{\varepsilon \leqslant u_1 < \dots < u_{k+1} \leqslant 1\}$ which is a semialgebraic subset of the unit simplex $\mathbf{\Delta}_{k+1} = \{0 \leqslant u_1 \leqslant \dots \leqslant u_{k+1} \leqslant 1\}$. So we need to study the asymptotics of $(k+1)! \int_{\{\varepsilon \leqslant u_1 < \dots < u_{k+1} \leqslant 1\}} Tr_{L^2}\left(e^{-u_1\Delta}V_1 \dots e^{-u_{k+1}\Delta}V_{k+1}\right) du_1 \dots du_{k+1}$. Setting $\chi = V_1 \boxtimes \dots \boxtimes V_{k+1} \in C^\infty(M^{k+1}, End(E)^{k+1})$ and using the notations and conventions from paragraphs 4.3.5 and 4.3.6, the blow–up from definition 4.9 yields a blow–down map

$$\begin{aligned}
\beta : (x, h_1, \dots, h_k, t_1, \dots, t_{k+1}) &\longmapsto ((x_1 = x, x_i = x + \sum_{j=1}^{i-1}(t_j \dots t_{k+1})h_j)_{i=2}^{k+1}, (u_l = \prod_{j=l}^{k+1} t_j^2)_{l=1}^{k+1}) \\
U \times \mathbb{R}^{kd} \times \Omega_\varepsilon &\longmapsto U^{k+1} \times \{\varepsilon \leqslant u_1 < \dots < u_{k+1} \leqslant 1\}
\end{aligned}$$

where Ω_ε is the **semialgebraic set** defined by $\Omega_\varepsilon = \{\varepsilon \leqslant (t_1 \dots t_{k+1})^2\} \cap [0,1]^{k+1}$. Now, following the calculations of paragraph 4.3.7 we set :

$$\omega(\chi) = \left(\int_{\mathbb{R}^{d(k+1)}} t_{k+1}^{2k+1-d} P(t_1, \dots, t_k)\beta^* J(t,x,h;\chi) d^d x d^{kd} h\right) dt_1 \wedge \dots \wedge dt_{k+1}$$

where $t_{k+1}^{d-2k-1}\omega$ is a smooth differential form of top degree on the cube $[0,1]^{k+1}$. To extract the singular part of $\int_{\Omega_\varepsilon}\omega$, we need to Taylor expand $\int_{\mathbb{R}^{d(k+1)}} A(t,x,h;\chi)d^d x d^{kd} h$ in the variable t_{k+1} :

$$\int_{\Omega_\varepsilon}\omega \simeq \sum_{j=0} \int_{\Omega_\varepsilon} \frac{t_{k+1}^{2k+1-d+j}}{j!} P(t_1, \dots, t_k) \underbrace{\left(\partial_{t_{k+1}}^j \int_{\mathbb{R}^{d(k+1)}} A(t,x,h;\chi) d^d x d^{kd} h\right)|_{t_{k+1}=0}} dt_1 \wedge \dots \wedge dt_{k+1}$$

where the term underbraced is a conormal distribution of $\chi \in C_c^\infty(U^{k+1}, End(E)^{\boxtimes k+1})$ supported by d_{k+1} by the results of paragraph 4.3.8. So setting $\chi = V^{\boxtimes k+1}$, we can view the term underbraced as a functional of V :

$$V \in C_c^\infty(U, End(E)) \mapsto \left(\partial_{t_{k+1}}^j \int_{\mathbb{R}^{d(k+1)}} A(t,x,h;V^{\boxtimes k+1}) d^d x d^{kd} h\right)|_{t_{k+1}=0}$$

is an element of $\mathcal{O}_{loc,k+1}\left(J^j E\right)$.

Then to extract precise asymptotics, set

$$\omega_j = \frac{t_{k+1}^{2k+1-d+j}}{j!}\left(\partial_{t_{k+1}}^j P \int_{\mathbb{R}^{d(k+1)}} A(t,x,h;\chi) d^d x d^{kd} h\right)|_{t_{k+1}=0} dt_1 \wedge \dots \wedge dt_{k+1}.$$

Then we may slice the **semialgebraic set** Ω_ε by the fibers $(t_1 \dots t_{k+1})^2 =$ constant of the map $F(t_1, \dots, t_{k+1}) = (t_1 \dots t_{k+1})^2$ which means we will push–forward the integral of ω_j along the fibers of the b-map $F : (t_1, \dots, t_{k+1}) \in [0,1]^{k+1} \mapsto F(t_1, \dots, t_{k+1}) \in \mathbb{R}$ [37, def 2.11

p. 16] [47, p. 51–52] where $[0,1]^{k+1}$ is viewed as a b-manifold in the sense of Melrose [37, def 2.2 p. 8] [47, p. 51]. Then we will conclude by using the pushforward Theorem of Melrose [47, Thm 4 p. 58] in the form discussed in the nice survey of Grieser [37, Thm 3.6 p. 25]. We define the form $\frac{\omega_j}{dF}$ which is called Gelfand–Leray form [75, Lemma 5.11 p. 123] and the function $J_j(t)=\int_{F=t}\frac{\omega_j}{dF}$. Then by Fubini's Theorem $\int_{\Omega_\varepsilon}\omega_j=\int_\varepsilon^1 J_j(t)dt$ and Lemma 6.2 implies each $\varepsilon\mapsto\int_{\Omega_\varepsilon}\omega_j$ admits an asymptotic expansion of the required form which concludes the proof. $\square$

6.1.1. *Pushforward Lemma.* Here we state the Lemma on asymptotic integrals used in the previous proposition.

Lemma 6.2 (Pushforward by Jeanquartier, Melrose)**.** *Let $\omega\in\Omega^n_c([0,1]^n)$ be a smooth differential form of top degree on $[0,1]^n$ and $F:(t_1,\dots,t_n)\in\mathbb{R}^n_+\mapsto t_1^2\dots t_n^2\in\mathbb{R}$. Then the map*

$$t\mapsto J(t)=\int_{F^{-1}(t)}\frac{t_n^{-m}\omega}{dF}=\left\langle\delta(t-F),t_n^{-m}\omega\right\rangle$$

*has an asymptotic expansion : $J(t)\sim\sum_{p,q}t^p\log(t)^q a_{p,q}(\omega)$ where $p\in\frac{\mathbb{Z}}{2}$ runs over a finite set of growing arithmetic sequences of rational numbers and $a_{p,q}$ is a **distribution** supported by the algebraic set $\{F=0\}$.*

This implies that the map $\varepsilon\longmapsto\int_\varepsilon^1 J(t)dt$ also has an asymptotic expansion : $\int_\varepsilon^1 J(t)dt\sim\sum_{p,q}\varepsilon^p\log(\varepsilon)^q b_{p,q}(\omega)$ where $p\in\frac{\mathbb{Z}}{2}$ runs over a finite set of growing arithmetic sequences of rational numbers and $b_{p,q}$ are distributions supported by $F=\{0\}$.

Proof. The result for smooth forms and real analytic F is due to Jeanquartier [27] [75, Theorem 5.54 p. 155]. Here we need the same result for a polyhomogeneous top form $t_n^{-m}\omega$ and $F=t_1^2\dots t_n^2$ which is a particular case of the pushforward Theorem of Melrose [37, Thm 3.6 p. 25] [47] by the b-map F which yields an index set contained in $\frac{\mathbb{Z}}{2}$ since the b-map F vanishes at order 2 on each boundary face of $[0,1]^n$. Let us give a proof based on remarks from Jeanquartier on the Mellin transform [28]. The index set of the asymptotics of $t\mapsto\langle\delta(t-F),t_n^{-m}\varphi\rangle$ is exactly given by the poles with multiplicity of the Mellin transform $\int_0^\infty t^sJ(t)\frac{dt}{t}=\int_{[0,1]^n}F^{s-1}t_n^{-m}\varphi d^nt$ by [28, Prop 4.3 p. 304 and Prop 4.4 p. 306]. By successive Taylor expansion with remainder as follows, start from φ then Taylor expand with remainder at order N in t_1 keeping other variables $(t_2,\dots,t_n)$ as parameters, then Taylor expanding successively in $t_2,\dots,t_n$ with remainder at order N yields : $\varphi(t_1,\dots,t_n)=\sum_{0\leqslant\alpha_1,\dots,\alpha_n\leqslant N}\prod t_i^{\alpha_i}c_\alpha$ where c_α depends on t_i iff $\alpha_i=N$. Then plugging under the integral yields that $s\mapsto\int_{[0,1]^n}F^{s-1}t_n^{-m}\varphi d^nt$ has analytic continuation as a meromorphic function on $\mathbb{C}$ with singular terms of the form $\left(\prod_{i=1}^{n-1}\frac{1}{2s+\alpha_i-1}\right)\frac{1}{2s+\alpha_n-1-m}$ hence poles are in $\{s\in\frac{1+m-\mathbb{N}}{2}\}$ with multiplicity at most n. $\square$

6.2. **Every $\mathcal{R}\det$ solution of problem 2.7 are obtained by local renormalization.** By Lemma 6.1, $\forall k\in\mathbb{N}$, $(V_1,\dots,V_{k+1})\mapsto FP|_{\varepsilon=0}Tr_{L^2}\left(e^{-2\varepsilon\Delta}\Delta^{-1}V_1\dots e^{-2\varepsilon\Delta}\Delta^{-1}V_{k+1}\right)$ is multilinear continuous hence it can be represented as a distributional pairing

$$FP|_{\varepsilon=0}Tr_{L^2}\left(e^{-2\varepsilon\Delta}\Delta^{-1}V_1\dots e^{-2\varepsilon\Delta}\Delta^{-1}V_{k+1}\right)=\langle\mathcal{R}t_{k+1},V_1\boxtimes\dots\boxtimes V_{k+1}\rangle$$

by the multilinear Schwartz kernel Theorem. Exactly as in the proof of subsubsection 4.3.4, we find that for $(V_1,\dots,V_{k+1})\in C^\infty(M,End(E))^{k+1}$ such that $\text{supp}(V_1)\cap\dots\cap\text{supp}(V_{k+1})=\emptyset$,

$$FP|_{\varepsilon=0}Tr_{L^2}\left(e^{-2\varepsilon\Delta}\Delta^{-1}V_1\dots e^{-2\varepsilon\Delta}\Delta^{-1}V_{k+1}\right)=Tr_{L^2}\left(\Delta^{-1}V_1\dots\Delta^{-1}V_{k+1}\right)$$

where the L^2 trace on the r.h.s is well–defined since $WF(\Delta^{-1}V_1)\cap\dots\cap WF(\Delta^{-1}V_{k+1})=\emptyset$. Therefore arguing as in subsubsection 4.3.4 we find that for $n\leqslant\frac{d}{2}$, $\mathcal{R}t_n$ is a distributional extension of $t_n=\mathbf{G}(x_1,x_2)\dots\mathbf{G}(x_n,x_1)$ and for $n>\frac{d}{2}$, the composition $e^{-2\varepsilon\Delta}\Delta^{-1}V_1\dots e^{-2\varepsilon\Delta}\Delta^{-1}V_{k+1}\in\Psi^{-2k}(M,E)$ hence of trace class [24, Prop B 20] uniformly in $\varepsilon\in(0,1]$ hence

$$FP|_{\varepsilon=0}Tr_{L^2}\left(e^{-2\varepsilon\Delta}\Delta^{-1}V_1\dots e^{-2\varepsilon\Delta}\Delta^{-1}V_{k+1}\right)=Tr_{L^2}\left(\Delta^{-1}V_1\dots\Delta^{-1}V_{k+1}\right)$$

where the r.h.s. is well–defined as in the case with zeta regularization.

Now let $P_{n,\varepsilon}\in\mathcal{O}_{loc}\otimes\mathbb{C}[\varepsilon^{-\frac{1}{2}},\log(\varepsilon)]$ from Lemma 6.1 s.t.

$$\lim_{\varepsilon\to0}Tr_{L^2}\left((e^{-2\varepsilon\Delta}\Delta^{-1}V)^n\right)-P_{n,\varepsilon}(V)=FP|_{\varepsilon=0}Tr_{L^2}\left((e^{-2\varepsilon\Delta}\Delta^{-1}V)^n\right).$$

One should think of $P_{n,\varepsilon}$ as being the **singular part** of $Tr_{L^2}\left(e^{-2\varepsilon\Delta}\Delta^{-1}V_1\dots e^{-2\varepsilon\Delta}\Delta^{-1}V_n\right)$. Then set $P_\varepsilon(V)=\sum_{n=1}^{\frac{d}{2}}P_{n,\varepsilon}(V)$, we have

$$\det{}_F\left(Id+e^{-2\varepsilon\Delta}\Delta^{-1}V\right)e^{-P_\varepsilon(V)}=\underbrace{\exp\left(\sum_{n=1}^{\frac{d}{2}}Tr_{L^2}\left((e^{-2\varepsilon\Delta}\Delta^{-1}V)^n\right)-P_{n,\varepsilon}(V)\right)}\underbrace{\det{}_{[\frac{d}{2}]+1}\left(Id+e^{-2\varepsilon\Delta}\Delta^{-1}V\right)}$$

by the factorization properties of Gohberg–Krein's determinant [66, d) Thm 9.2 p. 75]. The individual factors underbraced converge as follows :

- $\lim_{\varepsilon\to0^+}\det_{[\frac{d}{2}]+1}\left(Id+e^{-2\varepsilon\Delta}\Delta^{-1}V\right)=\det_{[\frac{d}{2}]+1}\left(Id+\Delta^{-1}V\right)$ because $e^{-2\varepsilon\Delta}\Delta^{-1}V\to\Delta^{-1}V\in\Psi^{-2}(M,E)$ hence in the Schatten ideal $\mathcal{I}_{[\frac{d}{2}]+1}$ and Gohberg–Krein's determinant $H\mapsto\det_{[\frac{d}{2}]+1}(Id+H)$ depends continuously on $H\in\mathcal{I}_{[\frac{d}{2}]+1}$.
- $\lim_{\varepsilon\to0^+}\exp\left(\sum_{n=1}^{\frac{d}{2}}Tr_{L^2}\left((e^{-2\varepsilon\Delta}\Delta^{-1}V)^n\right)-P_{n,\varepsilon}(V)\right)=\exp\left(\sum_{n=1}^{\frac{d}{2}}\left\langle\mathcal{R}t_n,V^{\boxtimes n}\right\rangle\right)$ where $\mathcal{R}t_n$ is a distributional extension of $t_n=\mathbf{G}(x_1,x_2)\dots\mathbf{G}(x_n,x_1)$ by construction.

Thus it is immediate that

$$\mathcal{R}\det(\Delta+V)=\lim_{\varepsilon\to0^+}\det{}_F\left(Id+e^{-2\varepsilon\Delta}\Delta^{-1}V\right)e^{-P_\varepsilon(V)}=\exp\left(\sum_{n=1}^{\frac{d}{2}}\left\langle\mathcal{R}t_n,V^{\boxtimes n}\right\rangle\right)\det{}_{[\frac{d}{2}]+1}\left(Id+\Delta^{-1}V\right)$$

hence it satisfies the representation formula 2.17 which makes it a solution of problem 2.7. If we are given any other solution $\mathcal{R}_2\det$ of problem 2.7, then by the free transitive action of $\mathcal{O}_{loc,[\frac{d}{2}]}$, we know that there exists $Q\in\mathcal{O}_{loc,[\frac{d}{2}]}$ s.t. $\mathcal{R}_2\det(\Delta+V)=e^{Q(V)}\mathcal{R}\det(\Delta+V)=\lim_{\varepsilon\to0^+}\det_F\left(Id+e^{-2\varepsilon\Delta}\Delta^{-1}V\right)e^{(Q-P_\varepsilon)(V)}$ which shows that $\mathcal{R}_2\det$ is obtained by renormalization by subtraction of local counterterms.

6.3. **Relation with Gaussian Free Fields.**

In the bosonic case, there is a nice interpretation of the renormalized determinants from Theorem 2 in terms of the Gaussian Free Field.

6.3.1. *Probabilistic representation.* We next briefly recall some probabilistic definition of the Gaussian Free Field (GFF) associated to our positive elliptic operator Δ which is represented as a random distribution on M.

Definition 6.3 (Bundle valued Gaussian Free Field)**.** *Under the geometric assumption from definition 2.1, if $\Delta\ : C^\infty(E)\mapsto C^\infty(E)$ is* ***positive, self–adjoint*** *then the Gaussian free field ϕ associated to Δ is defined as follows : denote by $(e_\lambda)_{\lambda\in\sigma(\Delta)}$ the spectral resolution associated to Δ. Consider a sequence $(c_\lambda)_{\lambda\in\sigma(\Delta)}, c_\lambda\in\mathcal{N}(0,1)$ of independent, identically distributed Gaussian random variables. Then we define the quantum field ϕ as the random series*

$$\phi=\sum_{\lambda\in\sigma(\Delta)}\frac{c_\lambda}{\sqrt{\lambda}}e_\lambda \tag{6.2}$$

where the sum runs over the eigenvalues of Δ and the series converges almost surely as distributional section in $\mathcal{D}'(M,E)$.

The covariance of the Gaussian free field defined above is the Green function :

$$\boxed{\mathbf{G}(x,y)=\sum_{\lambda\in\sigma(\Delta)}\frac{1}{\lambda}e_\lambda(x)\boxtimes e_\lambda(y)}$$

where the above series converges in $\mathcal{D}'(M\times M,E\boxtimes E)$.

A classical result characterizes the support of the functional measure :

Lemma 6.4 (Regularity of bundle GFF)**.** *Using the notations of definition 6.3, the random section ϕ converges almost surely in the Sobolev space $H^s(E)$ for every $s<1-\frac{d}{2}$.*

In Euclidean quantum field theory, there is an analogy between considering a discrete GFF on a lattice with spacing $\sqrt{\varepsilon}$, whose propagator is a discrete Green function which is the inverse of the discrete Laplacian and considering the heat regularized GFF $\phi_\varepsilon=e^{-\varepsilon\Delta}\phi$ whose covariance reads $e^{-2\varepsilon\Delta}\Delta^{-1}$. For discrete Laplacians Δ_ε on a regular lattice of mesh ε, there are beautiful results on the asymptotics of $\det(\Delta_\varepsilon)$ [10] (see [43] for related results) :

Theorem 5. *On the flat torus $\mathbb{T}^2$, for discrete Laplacian Δ_ε with mesh ε and denote by ϕ_ε the corresponding* ***discrete GFF****, if $V\in C^\infty(\mathbb{T}^2)$ s.t. $\int_{\mathbb{T}^2}V=0$ then :*

$$\frac{\det_\zeta(\Delta+V)}{\det_\zeta(\Delta)}=\lim_{\varepsilon\to0}\frac{\det(\Delta_\varepsilon+V)}{\det(\Delta_\varepsilon)}=\lim_{\varepsilon\to0^+}\mathbb{E}\left(e^{-\frac{1}{2}\int_{\mathbb{T}^2}V\phi_\varepsilon^2}\right)^{-2}. \tag{6.3}$$

In the bosonic case, replacing lattice regularization by the heat regularized GFF, we prove an analog of the above Theorem and describe all renormalized determinants from Theorem 2 as coming from the local renormalization of Gaussian free fields partition function as follows :

Theorem 6 (GFF representation)**.** *Under the assumptions of definition 6.3. Let ϕ be the Gaussian free field with covariance $\mathbf{G}$. Denote by $\phi_\varepsilon=e^{-\varepsilon\Delta}\phi$ the heat regularized GFF.*

Then a function $V\mapsto\mathcal{R}\det(\Delta+V)$ is a renormalized determinant in the sense of definition 2.7 if and only if there exists a sequence $(\Lambda_\varepsilon: C^\infty(E)\mapsto C^\infty(E))_{\varepsilon\in(0,1]}$ of ***smooth***

local *polynomial functionals of minimal degree such that the following limit exists :*

$$\mathcal{R}\det\left(Id+\Delta^{-1}V\right)^{-\frac{1}{2}}=\lim_{\varepsilon\to 0^+}\mathbb{E}\left(\exp\left(-\frac{1}{2}\int_M\langle\phi_\varepsilon,V\phi_\varepsilon\rangle-\Lambda_\varepsilon\left(V\right)(x)dv(x)\right)\right). \quad (6.4)$$

Furthermore, if $V\in C^\infty(End(E))$ defines a positive operator on $L^2(E)$, we denote by μ the Gaussian measure of covariance Δ^{-1} then the limit of measures

$$\nu=\lim_{\varepsilon\to 0^+}\frac{\exp\left(-\frac{1}{2}\int_M\langle\phi_\varepsilon,V\phi_\varepsilon\rangle-\Lambda_\varepsilon\left(V\right)(x)dv(x)\right)}{\mathbb{E}\left(\exp\left(-\frac{1}{2}\int_M\langle\phi_\varepsilon,V\phi_\varepsilon\rangle-\Lambda_\varepsilon\left(V\right)(x)dv(x)\right)\right)}\mu \quad (6.5)$$

exists as a Gaussian measure on $\mathcal{D}'(M)$ with covariance $(\Delta+V)^{-1}$ and ν is absolutely continuous w.r.t. μ iff $1\leqslant d\leqslant 3$ otherwise the measures (ν,μ) are mutually singular.

The intuitive idea is very simple, in QFT the renormalization problem arises from the fact that fields are irregular distributions then a natural idea is to study a regularized version of the field and see if one can perform an explicit renormalization of the partition function by subtracting **explicit local counterterms** in the action functional. Theorem 6 follows immediately from Theorem 2 once we reformulate the partition function $\mathbb{E}(e^{-\int_M\langle\varphi_\varepsilon,V\varphi_\varepsilon\rangle})$, where $\phi_\varepsilon=e^{-\varepsilon\Delta}\phi$ is the smeared GFF, in terms of Fredholm determinants $\det_F\left(I+\Delta^{-1}e^{-\varepsilon\Delta}Ve^{-\varepsilon\Delta}\right)$ which is the goal of the next paragraph. In a companion paper [13, Prop 1.4], we give a simple derivation of the above Theorem 6 using elementary commutator arguments when $\dim(M)\leqslant 4$.

6.3.2. *Fredholm determinants and partition functions.* The following Lemma relates partition functions and Fredholm determinants :

Lemma 6.5 (Field regularization.)**.** *Under the assumptions of definition 6.3, let $\phi_\varepsilon=e^{-\varepsilon\Delta}\phi$ be the mollified GFF.*

Then for every $\varepsilon>0$, the following relation holds true :

$$\mathbb{E}\left(\exp\left(-\frac{1}{2}\int_M\langle\phi_\varepsilon,V\phi_\varepsilon\rangle\,dv(x)\right)\right)=det_F\left(I+e^{-\varepsilon\Delta}\Delta^{-1}e^{-\varepsilon\Delta}V\right)^{-\frac{1}{2}}.$$

Proof. This is an immediate consequence of [34, Remark 1 p. 211] since the operator $(\Delta)^{-\frac{1}{2}}e^{-\varepsilon\Delta}Ve^{-\varepsilon\Delta}(\Delta)^{-\frac{1}{2}}$ is positive, self–adjoint and smoothing hence trace class on $L^2(E)$. □

6.4. **The renormalized functional measure.** In the previous part, we have constructed renormalized functional determinants to rigorously define the partition function. The following Proposition answers some natural questions about the corresponding renormalized functional measure.

Proposition 6.6. *Under the assumptions of definition 6.3, assume $V\in C^\infty(End(E))$ is Hermitian. Let μ denote the GFF measure on $\mathcal{D}'(M,E)$ with covariance $\mathbf{G}$ which is the Schwartz kernel of Δ^{-1}. Then there exists $P_\varepsilon(.)=\int_M\Lambda_\varepsilon\left(j^{d-3}.\right)\in\mathcal{O}_{loc,[\frac{d}{2}]}\left(J^{d-3}E\right)\otimes_{\mathbb{C}}\mathbb{C}[\varepsilon^{-\frac{1}{2}},\log(\varepsilon)]$ s.t. the limit*

$$\nu=\lim_{\varepsilon\to 0^+}\exp\left(-\frac{1}{2}\int_M\left(\langle\phi_\varepsilon,V\phi_\varepsilon\rangle-\Lambda_\varepsilon(j^{d-3}V)\right)dv(x)\right)\mu$$

converges to a Gaussian measure on $\mathcal{D}'(M,E)$ which is ***absolutely continuous*** *w.r.t. μ if $d=(2,3)$ and the measure (μ,ν)* ***are mutually singular*** *when $d\geqslant 4$.*

Proof. Define $\nu_\varepsilon = \frac{\exp\left(-\frac{1}{2}\int_M\left(\langle\phi_\varepsilon,V\phi_\varepsilon\rangle-\Lambda_\varepsilon(j^{d-3}V)\right)dv(x)\right)}{\mathbb{E}\left(\exp\left(-\frac{1}{2}\int_M\left(\langle\phi_\varepsilon,V\phi_\varepsilon\rangle-\Lambda_\varepsilon(j^{d-3}V)\right)dv(x)\right)\right)}\mu$ for $\varepsilon>0$. This is a Gaussian measure whose covariance is $\left(\Delta+e^{-\varepsilon\Delta}Ve^{-\varepsilon\Delta}\right)^{-1}$ by [34, Prop 9.3.2 p. 213]. When $\varepsilon\to 0^+$, this covariance converges to $(\Delta+V)^{-1}$ as **bilinear forms on** $C^\infty(M)\times C^\infty(M)$ for the weak topology [34, iv) p. 208] since $e^{-\varepsilon\Delta}\to Id$ in the **strong operator topology** when $\varepsilon\to 0^+$. A necessary and sufficient condition for the renormalized measure to be absolutely continuous w.r.t. the initial measure is given by a Theorem of Shale [67, Thm I.23 p. 41] is that $\Delta^{-\frac{1}{2}}V\Delta^{-\frac{1}{2}}\in\Psi^{-2}(M,E)$ is Hilbert–Schmidt which holds true only if $\dim(M)=d\leqslant 3$. $\square$

7. Quillen's determinant line bundle.

We recall the definition of Quillen's determinant line bundle which is an adaptation of the definition of Segal [65, p. 137-138], Furutani [31] and Melrose–Rochon [48] where holomorphicity properties are manifest. The reader can also look at [63, section 5.3 p. 642] for a very nice account of determinant line bundles for families of Ψdos.

Definition 7.1 (Quillen's universal determinant line bundle)**.** *Using the notations of subsubsection 1.0.3. Recall $\mathcal{I}_1(\mathcal{H})$ denotes the ideal of trace class operators on some Hilbert space $\mathcal{H}$. Let $(T_b)_{b\in B}$ be a holomorphic family of Fredholm operators from $\mathcal{H}_0\mapsto\mathcal{H}_1$ of index 0, parametrized by a complex Banach manifold B. Consider the bundle*

$$\mathcal{G}=\bigcup_{b\in B}T_b(Id+\mathcal{I}_1(\mathcal{H}_1))\simeq B\times(Id+\mathcal{I}_1(\mathcal{H}_1))$$

which fibers over the complex Banach manifold B.

Then we define the determinant line bundle ***Det*** *$\mapsto B$ to be the quotient $\mathcal{G}\times\mathbb{C}/\sim$ where $(A(I+T),z)\sim(A,\det_F(I+T)z)$. The canonical section $\underline{\det}(T)$ is defined to be the equivalence class $T\mapsto[T,1]$.*

This definition is functorial since it works for any holomorphic family $(T_b)_{b\in B}$ and holomorphicity is checked as in the work of Furutani [31]. Quillen's line bundle is recovered by letting B to be the space $\mathbf{Fred}_0(\mathcal{H}_0,\mathcal{H}_1)$ of Fredholm operators of index 0 as proved by Furutani [31, section 2 and prop 2.1]. Let use recall that

Lemma 7.2. *The canonical section $T\mapsto\underline{\det}(T)=[T,1]$ vanishes if and only if T non invertible.*

Proof. $[T,1]\simeq[\tilde{T},0]$ means there exists $I+A, A\in\mathcal{I}^1$ s.t. $\tilde{T}(I+A)=T$ and $\det_F(I+A)=0$ hence $I+A$ is non invertible and so is T. Conversely, if T non invertible there is a finite rank operator t such that $T+t$ invertible since T is Fredholm of index 0. Therefore $T=(T+t)(I-(T+t)^{-1}t)$ where $(I-(T+t)^{-1}t)$ is in the determinant class and $(I-(T+t)^{-1}t)$ non invertible. Finally $[T,1]\sim[T+t,\det_F((I-(T+t)^{-1}t))]=0$. $\square$

8. Proof of Theorem 3.

We follow the notations from subsubsection 1.0.3. The way Quillen trivializes the line bundle is by constructing a smooth Hermitian metric on $\mathcal{L}$ named Quillen's metric and calculate explicitely the curvature of the corresponding Chern connection which is exactly the Kähler form on $\mathcal{A}$. Then he shows that by modifying the Hermitian metric, one can produce a modified Chern connection ∇ which is flat. It follows from the contractibility of $\mathcal{A}$ that flat sections for ∇ trivialize $\mathcal{L}$ **holomorphically**. Here the setting is slightly different and our approach to holomorphic trivialization is more direct and does not use Quillen metrics. We already know that the canonical section $\iota^*\underline{\det}$ has the same divisor on $\mathcal{A}$ as any solution $\mathcal{R}\det$ of Theorem 2. Hence the ratio $\frac{\iota^*\underline{\det}(D)}{\mathcal{R}\det(D)}$ is holomorphic without zeros on $\mathcal{A}$. It remains to show that this is locally bounded in order to conclude that this is a holomorphic section without zeros by proposition 3.6, hence it yields a holomorphic trivialization of $\mathcal{L}$.

Following Segal and Furutani, we define open sets U_t indexed by finite rank operators t such that $U_t = \{T \in \mathbf{Fred}_0(\mathcal{H})$ s.t. $T + t$ invertible$\}$. Since elements in $\mathbf{Fred}_0(\mathcal{H})$ have Fredholm index 0, the collection $(U_t)_t$ forms an open cover of $\mathbf{Fred}_0(\mathcal{H})$. Then we trivialize $\mathcal{L}$ over U_t by the never vanishing section $T \in U_t \mapsto [T+t, 1]$ which is holomorphic by the proof of Furutani. In the local trivialization, the canonical section $T \mapsto \underline{\det(T)} = [T, 1]$ is identified with the holomorphic function $\det_F(I-(T+t)^{-1}t)$ since $[T, 1] \sim [T+t, \det_F(I-(T+t)^{-1}t)] = \det_F(I-(T+t)^{-1}t)[T+t, 1]$.

Now we shall prove a technical

Lemma 8.1. *Let T_0 be an invertible operator in $\iota(\mathcal{A})$ such that for all $T \in \iota(\mathcal{A})$, $T - T_0$ is in the Schatten ideal $\mathcal{I}_{[\frac{d}{k}]+1}$, $k = (1, 2)$.*

It follows that for $p = [\frac{d}{k}] + 1$, Gohberg–Krein's determinant $\det_p\left(I + T_0^{-1}(T - T_0)\right)$ is holomorphic on $\iota(\mathcal{A})$. Then the section

$$T \in \iota(\mathcal{A}) \longmapsto det_p\left(I + T_0^{-1}(T - T_0)\right)^{-1} \underline{\det(T)} \tag{8.1}$$

defines a global holomorphic section of $\mathbf{Det} \mapsto \iota(\mathcal{A})$ which never vanishes.

Proof. It suffices to prove the claim on each open subset $U_t \cap \iota(\mathcal{A})$ where the canonical section $T \mapsto \det(T)$ is identified with $T \in U_t \mapsto \det_F(I - (T+t)^{-1}t)$ by the local trivialization.

Use the identity $I + T_0^{-1}(T - T_0) = T_0^{-1}T$ and $I - (T+t)^{-1}t = (T+t)^{-1}T$. By the multiplicativity of Fredholm determinants, for every invertible $T \in U_t \cap \iota(\mathcal{A})$, we find that

$$\begin{aligned}
& \det{}_F(I - (T+t)^{-1}t)\det{}_p\left(I + T_0^{-1}(T - T_0)\right)^{-1} \\
= \ & \det{}_F(I - (T+t)^{-1}t)\det{}_F\left(I + R_p\left(T_0^{-1}(T - T_0)\right)\right)^{-1} \\
= \ & \det{}_F\left((T+t)^{-1}T_0(I + T_0^{-1}(T - T_0))(I + R_p(T_0^{-1}(T - T_0)))^{-1}\right).
\end{aligned}$$

For every $T \in U_t \cap \iota(\mathcal{A})$, the operator $(T+t)^{-1}T_0$ is invertible. By the spectral mapping Theorem for entire functions, the singular value $\lambda_k(I + T_0^{-1}(T - T_0))(I + R_p(T_0^{-1}(T - T_0)))^{-1} = (1+\lambda_k)(1+\lambda_k)^{-1}\exp\left(-\sum_{j=1}^{p-1}\frac{(-1)^j}{j}\lambda_k^j\right) = \exp\left(-\sum_{j=1}^{p-1}\frac{(-1)^j}{j}\lambda_k^j\right) \neq 0$ where $\lambda_k \in \sigma\left(T_0^{-1}(T - T_0)\right)$ since $T_0^{-1}(T - T_0) \in \mathcal{I}_{[\frac{d}{k}]+1}$ is a compact operator. It follows that

$0 \notin \sigma(I + T_0^{-1}(T - T_0))(I + R_p(T_0^{-1}(T - T_0)))^{-1}$ and the operator $(I + T_0^{-1}(T - T_0))(I + R_p(T_0^{-1}(T - T_0)))^{-1}$ is invertible.

Finally $(T + t)^{-1}T_0(I + T_0^{-1}(T - T_0))(I + R_p(T_0^{-1}(T - T_0)))^{-1}$ is invertible for every $T \in U_t \cap \iota(\mathcal{A})$ and in the determinant class, hence its Fredholm determinant never vanishes. It follows that $T \in U_t \cap \iota(\mathcal{A}) \cap \text{invertible} \mapsto \det_F(I - (T + t)^{-1}t)\det_p\left(I + T_0^{-1}(T - T_0)\right)^{-1}$ extends uniquely as a never vanishing entire function on $U_t \cap \iota(\mathcal{A})$. □

Lemma 8.1 says the ratio $P + \mathcal{V} \in \mathcal{A} \mapsto \det_{[\frac{d}{k}]+1}(I + P^{-1}\mathcal{V})^{-1}\iota^*\underline{\det}(P + \mathcal{V})$ never vanishes over $\mathcal{A}$. Furthermore Corollary 2.9 states that $\mathcal{R}\det(P + \mathcal{V}) = \exp(g(\mathcal{V}))\det_{[\frac{d}{k}]+1}(I + P^{-1}\mathcal{V})$ where g is a polynomial function, therefore $\exp(g(\mathcal{V}))$ never vanishes and the holomorphic section $\sigma : P + \mathcal{V} \in \mathcal{A} \mapsto \mathcal{R}\det(P + \mathcal{V})^{-1}\iota^*\underline{\det}(P + \mathcal{V})$ never vanishes over $\mathcal{A}$ and defines a holomorphic trivialization of $\mathcal{L} : \tau : \mathcal{O}(\mathcal{L}) \mapsto Hol(\mathcal{A})$ such that the canonical section $\iota^*\underline{\det}(T)$ is sent to the entire function $T \in \mathcal{A} \mapsto \mathcal{R}\det(T)$. The second claim follows from the action of the renormalization group as in Theorem 2. Finally, every non vanishing section σ defines canonically a flat connection ∇ whose flat section is σ.

9. Appendix.

We give the proof of the following

Lemma 9.1. *Let P be a continuous polynomial function on $C^\infty(M)$ such that P is local in the sense*

$$\delta^2 P(w; u, v) = 0 \tag{9.1}$$

when (u, v) have disjoint supports and the linear term of P is given by integration against a smooth function. If $WF\left(\delta^2 P(V)\right) \subset N^*\left(d_2 \subset M^2\right)$ for all $V \in C^\infty(M)$ then $P \in \mathcal{O}_{loc}(C^\infty(M))$.

Proof. Equation 9.1 implies that all functional derivatives $\delta^n P$ of P have their Schwartz kernel supported on the deepest diagonal $d_n \subset M^n$ by [6, Proposition V.5] and that P is **additive** in the sense of [6]. Since P is a polynomial function, it equals its Taylor expansion $P(V) = \sum_{n=1}^{\deg(P)} P_n(V)$ where P_n homogeneous of degree n.

The smoothness condition on the linear term in P together with the microlocal condition on $\delta^2 P$ imply that δP is represented by integration against smooth function.

Therefore by uniqueness of the Taylor expansion each P_n satisfies equation 9.1. Let $\tilde{P}_n$ be the multilinear map corresponding to P_n that we freely identify with its Schwartz kernel $\tilde{P}_n$ by the kernel Theorem [6]. Now by a Theorem of Laurent Schwartz, the Schwartz kernel $\tilde{P}_n$ which is a distribution carried by the deepest diagonal has an expression in local coordinates $(x_1, \dots, x_n)$ in U^n as

$$\tilde{P}_n = \sum f_{[\alpha]}(x_1)\partial_{x_2}^{\alpha_2} \dots \partial_{x_n}^{\alpha_n}\partial_{x_1}^{\alpha_1}\delta_{\{0\}}^{\mathbb{R}^{d(n-1)}}(x_1 - x_2, \dots, x_1 - x_n)$$

where the sum is finite and $f_{[\alpha]}$ is a distribution in the variable x_1. It follows that the Schwartz kernel of the second derivative has the representation in local coordinates

$$\delta^2 P(V, x, y) = \sum f_{[\alpha]}(x)\partial_x^{\alpha_2} V(x) \dots \partial_x^{\alpha_n} V(x)\partial_y^{\alpha_1}\delta^{x-y}$$

which implies P satisfies condition 2 of [6, Lemma VI.9]. By [6, Lemma VI.9], this means $V \mapsto \nabla P_V$ is smooth. To summarize, P is additive, its differential δP_V is represented by integration against a smooth function ∇P_V and $V \mapsto \nabla P_V$ is smooth hence by [6, Theorem I.2], $P \in \mathcal{O}_{loc}(C^\infty(M))$. $\square$

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