Nonlinear Wightman fields — a poster

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The Hadamard product may be exploited to construct a nonlinear analog of a Wightman field.

LINEAR WIGHTMAN FIELDS

Free quantum fields, which are the only known Wightman fields in 3+1-dimensions, can be presented as creation and annihilation algebras, in the familiar terms of operator-valued distributions and for the special case of the massive scalar field, as

$$\begin{aligned} [a(k), a^\dagger(k')] &= (2\pi)^4\delta(k-k')2\pi\delta(k\cdot k-m^2)\theta(k_0), \\ \hat{\phi}(x) &= \int \left[a(k)\mathrm{e}^{\mathrm{i}k\cdot x} + a^\dagger(k)\mathrm{e}^{-\mathrm{i}k\cdot x}\right]\frac{\mathrm{d}^4k}{(2\pi)^4}, \end{aligned}$$

together with the trivial action of the annihilation operators on the vacuum state, $a(k)\left|0\right\rangle = 0$. In terms of operators, obtained by "smearing" the operator-valued distributions with "test functions" from a well-behaved function space, and constructing the vacuum state explicitly using a matrix permanent,

$$\begin{aligned} a_f^\dagger &= \int a^\dagger(k)\tilde{f}(k)\frac{\mathrm{d}^4k}{(2\pi)^4}, \qquad \hat{\phi}_f = a_{f^*} + a_f^\dagger, \\ [a_f, a_g^\dagger] &= (f,g) \\ &= \int \tilde{f}^*(k)2\pi\delta(k\cdot k-m^2)\theta(k_0)\tilde{g}(k)\frac{\mathrm{d}^4k}{(2\pi)^4}, \end{aligned}$$

$$\left\langle 0\right| a_{f_1}\cdots a_{f_m}a_{g_1}^\dagger\cdots a_{g_n}^\dagger\left|0\right\rangle = \delta_{m,n}\mathsf{Per}[(f_i,g_j)].$$

Lorentz invariance is ensured by tensor notation, whereas translation invariance requires that the positive semi-definite inner product (f,g) is diagonal in the wave-number basis; the spectrum condition requires the restriction to the forward light-cone that is ensured by the factor $2\pi\delta(k\cdot k-m^2)\theta(k_0)$; hermiticity requires $[a_f, a_g^\dagger]^\dagger = (f,g)^* = (g,f) = [a_g, a_f^\dagger]$; locality requires that $[\hat{\phi}_f, \hat{\phi}_g] = 0$ when f and g have space-like separated supports, which is ensured by $(f^*,g) = (g^*,f)$ in that case; the construction of a Hilbert space of vector states requires that the matrix (f_i,f_j) is positive semi-definite, for any set of test functions f_i, which is ensured by (f,g) being a positive semi-definite inner product on the test function space, so that (f_i,f_j) is a Gram matrix; and cluster decomposition is satisfied because of the algebraic properties of the commutator and its decrease at large space-like separation.

NONLINEAR WIGHTMAN FIELDS

All the above conditions are equally satisfied if we construct a *nonlinear Wightman field*, for which the simplest example is

$$[\mathbf{a}_f, \mathbf{a}_g^\dagger] = (\!(f,g)\!) = (f,g) + (f,g)^2, \quad \hat{\boldsymbol{\xi}}_f = \mathbf{a}_{f^*} + \mathbf{a}_f^\dagger,$$

$$\left\langle 0\right| \mathbf{a}_{f_1}\cdots \mathbf{a}_{f_m}\mathbf{a}_{g_1}^\dagger\cdots \mathbf{a}_{g_n}^\dagger\left|0\right\rangle = \delta_{m,n}\mathsf{Per}[(\!(f_i,g_j)\!)].$$

where the $(f_i,f_j)^2$ component of $(\!(f_i,f_j)\!)$ is a positive semi-definite matrix because it is a Hadamard product of positive semi-definite matrices, even though $(\!(f,g)\!)$ is not a positive semi-definite inner product. Arbitrary positively weighted sums of Hadamard products of (f_i,f_j) may be used to construct $(\!(f_i,f_j)\!)$, and several more elaborate constructions are described in [1]. Two additional conditions that are ordinarily required of a Wightman field are that (1) the quantum field is an operator-valued distribution, and (2) the Hilbert space supports a representation of the Poincaré group; the first of which is not satisfied because linearity has been given up, the latter of which either is not satisfied because we construct a Hilbert space using only a finite or countable number of test functions or else we construct a Hilbert space that supports a representation of the Poincaré group but does not have a countable basis, because in general we cannot as in the linear case write $\hat{\boldsymbol{\xi}}_{\lambda f+\mu g}$ as a linear combination $\lambda\hat{\boldsymbol{\xi}}_f + \mu\hat{\boldsymbol{\xi}}_g$. We note, however, that the use of a finite basis is a commonplace for practical modeling in Physics.

DISCUSSION

Linearity is fundamental to probability, but it is unthinkable as a constraint in classical field theory; linearity is also not satisfied when using real-space renormalization, which in general constructs higher-level observables as nonlinear functions of lower-level observables; nor is linearity satisfied when using effective field theories, for which parameters of the equations satisfied by observables may be arbitrary functions of a summary scale of the level of experimental detail. The "level of experimental detail" is determined for a Wightman field observable $\hat{\phi}_f$ or $\hat{\boldsymbol{\xi}}_f$ by a summary property of the test function f; the construction here preserves the linearity of the action of operators on a Hilbert space that is required for an algebra to generate probabilities, but gives up the linearity of the map of test functions into the algebra, thereby opening up an extensive class of nonlinear Wightman field models, as is described in some detail in [1], which brings a fresh perspective for comparison with Lagrangian and related approaches to QFT.

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[1] P. Morgan, arXiv:1211.2831 [math-ph].