THE WELL-POSEDNESS OF THE CAUCHY PROBLEM FOR THE DIRAC OPERATOR ON GLOBALLY HYPERBOLIC MANIFOLDS WITH TIMELIKE BOUNDARY

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Abstract. We consider the Dirac operator on globally hyperbolic manifolds with timelike boundary and show well-posedness of the Cauchy initial-boundary value problem coupled to MIT-boundary conditions. This is achieved by transforming the problem locally into a symmetric positive hyperbolic system, proving existence and uniqueness of weak solutions and then using local methods developed by Lax, Phillips and Rauch, Massey to show smoothness of the solutions. Our proof actually works for a slightly more general class of local boundary conditions.

1. Introduction

The well-posedness of the Cauchy problem for the Dirac operator on a Lorentzian manifold $(\widetilde{\mathcal{M}}, g)$ is a classical problem which has been exhaustively studied in many contexts. If the underlying background is *globally hyperbolic* a complete answer is known: In [17] it was shown that a fundamental solution for the Dirac equation can obtained from a fundamental solution of the conformal wave operator via the Lichnerowicz formula. Since the Cauchy problem for the conformal wave operator is well-posed [10, 32], it follows that the Cauchy problem for the Dirac operator is well-posed too. Even if there exists a plethora of models in physics where globally hyperbolic spacetimes has been used as background, there also exist substential situations which require a manifold to have a non-empty boundary. Indeed, recent developments in quantum field theory focused their attention on manifolds with timelike boundary [5, 36], e.g. anti-de Sitter spacetime [12, 13] and BTZ spacetime [7]. Moreover, experimental setups for studying the Casimir effect enclose (quantum) fields between walls, which may be mathematically described by introducing timelike boundaries [15]. In these settings, the correspondence between the well-posedness of the conformal wave operator and the Dirac operator breaks down due to the boundary condition. Even if the Cauchy problem for the conformal wave operator is proved to be well-posed in a large class of boundary conditions for stationary spacetimes [11], it is still not clear how to relate this with the Cauchy problem for the Dirac operator. In fact, solvability in the Dirac case very much depends on the boundary condition, e.g. if the Dirichlet boundary condition is applied to spinors on the boundary, in general there does not exists any smooth solution to the Dirac equation.
The goal of this paper is to investigate the well-posedness of the Cauchy problem for the Dirac operator in globally hyperbolic manifolds with timelike boundary in

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the following sense: Let $(\widetilde{\mathcal{M}}, g)$ be a globally hyperbolic spin manifold of dimension $n+1$. Globally hyperbolic as in [3, 27] means that the manifold admits a Cauchy surface, i.e. a hypersurface that is hit exactly once by every inextendible timelike curve. Moreover, it can always be chosen to be smooth and spacelike, see [6]. Without always mentioning it, we always assume that our manifolds are orientable. Note that for $n+1=4$, the case relevant for physics, globally hyperbolic manifolds are automatically spin. Let $\mathcal{N}$ be a submanifold of $\widetilde{\mathcal{M}}$ that with the induced metric is itself globally hyperbolic. Let $\widetilde{\Sigma}$ be a smooth spacelike Cauchy surface of $\widetilde{\mathcal{M}}$. Then, $\widehat{\Sigma} := \widetilde{\Sigma} \cap \mathcal{N}$ is a spacelike Cauchy surface for $\mathcal{N}$. We assume that $\mathcal{N}$ divides $\widetilde{\mathcal{M}}$ into two connected components. The closure of one of them we denote by $\mathcal{M}$ and refer to as a *globally hyperbolic manifold with timelike boundary.* On $\widetilde{\mathcal{M}}$ we choose a Cauchy time function $t\colon \widetilde{\mathcal{M}} \to \mathbb{R}$. Then $\{t^{-1}(s)\}_{s\in\mathbb{R}}$ gives a foliation by Cauchy surfaces and we set $\Sigma_s := t^{-1}(s) \cap \mathcal{M}$.

We always assume that we fix the spin structure on $\mathcal{M}$. Let $S\mathcal{M}$ denote the spinor bundle over $\mathcal{M}$ and $S\Sigma_0$ the induced spinor bundle over Σ_0. Moreover, let D be the Dirac operator on $S\mathcal{M}$, for details see Section 2.

Theorem 1. *The Cauchy problem for the Dirac operator with MIT-boundary condition on a globally hyperbolic spin manifold $\mathcal{M}$ with timelike boundary $\partial\mathcal{M}$ is well-posed i.e. for any $f \in \Gamma_{cc}(S\mathcal{M})$ and $h \in \Gamma_{cc}(S\Sigma_0)$ there exists a unique spacelike compact smooth solution ψ to the mixed initial-boundary value problem*

$$\begin{cases} \mathsf{D}\psi = f \\ \psi|_{\Sigma_0} = h \\ (\gamma(\mathtt{n}) - i)\psi|_{\partial\mathcal{M}} = 0 \end{cases} \tag{1.1}$$

which depends continuously on the data (f, h). Here the $\gamma(\mathtt{n})$ denotes Clifford multiplication with $\mathtt{n}$ the outward unit normal on $\partial\mathcal{M}$ and $\Gamma_{cc}(.)$ denotes the space of sections that are compactly supported in the interior of the underlying manifold.

For a subclass of stationary spacetimes with timelike boundary admitting a suitable timelike Killing vector field Theorem 1 was already proven in [23].

The *MIT boundary condition*, was introduced for the first time in [9] in order to reproduce the confinement of quark in a finite region of space: "Dirac waves" are indeed reflected on the boundary. Few years later, it was used in the description of hadronic states like baryons [8] and mesons [29]. More recently, the MIT boundary condition were employed in [35] for computing the Casimir energy in a a three-dimensional rectangular box, in [20] and [23, 24] in order to construct an integral representation for the Dirac propagator in Kerr-Newman and Kerr Black Hole Geometry respectively, and in [30] for proving the asymptotic completeness for linear massive Dirac fields on the Schwarzschild Anti-de Sitter spacetime.

Remark 2. Theorem 1 holds under more general local boundary conditions, namely for any linear non-invertible map $\mathsf{M}\colon \Gamma(S\partial\mathcal{M}) \to \Gamma(S\partial\mathcal{M})$ with constant kernel dimension and such that $\mathsf{M}\psi|_{\partial\mathcal{M}} = 0$ implies

$$\langle \psi \,|\, \gamma(e_0)\gamma(\mathtt{n})\psi \rangle_q = 0 \tag{1.2}$$

for all $q \in \partial\mathcal{M}$, see also Remark 24. Here e_0 is the globally defined unit timelike vector field defined by (2.1).

As we will see, the well-posedness of the Cauchy problem guarantees the existence of *Green operators* for the Dirac operator. In globally hyperbolic spin manifold with empty boundary, those operators play a role in the quantization of linear field theory: Indeed they fully characterize the space of solutions of the Dirac equation [14, 17], they implement the canonical anticommutation relations typical of any fermionic quantum field theory [2, 4, 16], and their difference, dubbed the casual propagator or Pauli-Jordan commutator, can be used to construct quantum states [19, 21, 22].

Remark 3. The Cauchy problem (1.1) is still well-posed in a larger class of initial data. But then some compatibility condition for $\mathfrak{f}$ and $\mathfrak{h}$ on $\partial\Sigma_0$ is needed—see Remark 20. In the subclass of stationary spacetimes considered in [23], as mentioned above, this compatibility condition reduces to the one therein. Without these compatibility conditions the solution would still exist but the singularities contained in a neighborhood of $\partial\Sigma_0$ would propagate with time along lightlike geodesics. In case, they hit again the boundary some boundary phenomena as reflection will occur. For future work it is of course interesting to obtain an explicit method to construct the corresponding Green operators and to obtain more information on how singularities behave when hitting the boundary.

Our strategy to prove the well-posedness of the Cauchy problem is as follows: First we restrict to a strip of finite time and prove general properties as finite propagation of speed and uniqueness in Section 3.1. Then using the theory of symmetric positive hyperbolic systems, see e.g. [26], to prove the existence of a weak solution in Section 3.2. Then in Section 3.3 we localize the problem by introducing suitable coordinates small enough that we can associate to our Dirac problem a hyperbolic system that fits into the class considered in [31, 34] and thus provides smoothness of the solution. With the help of uniqueness we can then easily glue together local solutions to obtain global ones in Section 4.

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2. Preliminaries

2.1. **Basis notations.** In the following we always assume that the spin structure is fixed. We denote by $S\mathcal{M}$ the associated *spinor bundle*, i.e. a complex vector bundle with $N := 2^{\lfloor \frac{n+1}{2} \rfloor}$-dimensional fibers, denoted by $S_p\mathcal{M}$ for $p \in \mathcal{M}$, fiberwise endowed with the canonical scalar product on $\mathbb{C}^N$

$$\langle \cdot \,|\, \cdot \rangle \colon S_p\mathcal{M} \times S_p\mathcal{M} \to \mathbb{C}$$

and with a *Clifford multiplication*, i.e. a fiber-preserving map $\gamma\colon T\mathcal{M} \to \mathrm{End}(S\mathcal{M})$. We denote the by $\Gamma_c(S\mathcal{M})$, $\Gamma_{cc}(S\mathcal{M})$, $\Gamma_{sc}(S\mathcal{M})$ resp. $\Gamma(S\mathcal{M})$ the spaces of compactly supported, compactly supported in the interior, spacelike compactly supported resp. smooth sections of the spinor bundle. The *(classical) Dirac operator* $\mathsf{D}\colon \Gamma(S\mathcal{M}) \to \Gamma(S\mathcal{M})$ is defined as the composition of the connection on $S\mathcal{M}$, obtained as a lift of the Levi-Civita connection ∇ on $T\mathcal{M}$, and the Clifford multiplication. Thus, in local coordinates this reads as

$$\mathsf{D} = \sum_{\mu=0}^{n} \imath\gamma(e_\mu)\nabla_{e_\mu}$$

where $(e_\mu)_{\mu=0,\ldots,n}$ is a local Lorentzian-orthonormal frame of $T\mathcal{M}$ and the Clifford multiplication satisfies $\gamma(u)\gamma(v)+\gamma(v)\gamma(u)=-2g(u,v)$ for every $u,v\in T_p\mathcal{M}$ and $p\in\mathcal{M}$.

Recall that after choice of the Cauchy time function t the metric on the globally hyperbolic manifold $\widetilde{\mathcal{M}}$ can be written as $g=g|_{\Sigma_t}-\beta^2dt^2$ where $\beta\colon\widetilde{\mathcal{M}}\to\mathbb{R}$ is a positive smooth function. Thus, although the local orthonormal frame e_μ in general only exists locally we have a globally defined unit timelike vector field

$$e_0:=\frac{1}{\beta}\partial_t \tag{2.1}$$

that we will use in the following. Note that in particular we have $\gamma(e_0)^{-1}=\gamma(e_0)$.

2.2. **Reformulation as a symmetric positive hyperbolic system.** In this section we will formulate the Dirac equation (1.1) as a symmetric positive hyperbolic system. For that we shortly recall the basic definition, for more details see [25, 26].

For the following definition, let $E\to\mathcal{M}$ be a real or complex vector bundle with finite rank N endowed with a fiberwise metric $\langle\cdot\,|\,\cdot\rangle$.

Definition 4. A linear differential operator $\mathfrak{L}\colon\Gamma(E)\to\Gamma(E)$ of first order is called a *symmetric system* over $\mathcal{M}$ if

(S) the principal symbol $\sigma_{\mathfrak{L}}(\xi)\colon E_p\to E_p$ is Hermitian with respect to $\langle\cdot\,|\,\cdot\rangle$ for every $\xi\in T_p^*\mathcal{M}$ and for every $p\in\mathcal{M}$.

Additionally, we say that $\mathfrak{L}$ is *positive* respectively *hyperbolic* if it holds:

(P) The bilinear form $\langle\,(\mathfrak{L}+\mathfrak{L}^\dagger)\cdot\,|\,\cdot\rangle$ on E_p is positive definite, where $\mathfrak{L}^\dagger$ denote the formal adjoint of $\mathfrak{L}$ with respect to $\langle\cdot\,|\,\cdot\rangle$.

(H) For every future-directed timelike covector $\tau\in T_p^*\mathcal{M}$, the bilinear form $\langle\sigma_{\mathfrak{L}}(\tau)\cdot\,|\,\cdot\rangle$ is positive definite on E_p.

Let us recall that for a first-order linear operator $\mathfrak{L}\colon\Gamma(E)\to\Gamma(E)$ the principal symbol $\sigma_{\mathfrak{L}}\colon T^*\mathcal{M}\to\mathrm{End}(E)$ can be characterized by $\mathfrak{L}(fu)=f\mathfrak{L}u+\sigma_{\mathfrak{L}}(df)u$ where $u\in\Gamma(\mathcal{M},E)$ and $f\in C^\infty(\mathcal{M})$. If we choose local coordinates on $\mathcal{M}$ and a local trivialization of E, any linear differential operator $\mathfrak{L}\colon\Gamma(E)\to\Gamma(E)$ of first order reads as

$$\mathfrak{L}:=A_0(p)\partial_t+\sum_{j=i}^{n}A_j(p)\partial_{x^j}+B(p)$$

where the the coefficient of A_0,A_j,B are $N\times N$ matrices, with N being the rank of E, depending smoothly on $p\in\mathcal{M}$. In local coordinates, Condition (S) in Definition 4 reduces to

$$A_0=A_0^\dagger\qquad\text{and}\qquad A_j=A_j^\dagger$$

for $j=1,\ldots,n$. Condition (P) reads as

$$\kappa+\kappa^\dagger>0\qquad\text{with}\quad\kappa:=B-\partial_tA_0-\sum_{j=1}^{n}\partial_{x^j}A_j\,,$$

while condition (H) can be stated as follow: For any covector $\tau=dt+\sum_j\alpha_jdx^j$,

$$\sigma_{\mathfrak{L}}(\tau)=A_0+\sum_{j=1}^{N-1}\alpha_jA_j\qquad\text{is positive definite.}$$

Remark 5. With this definition, we can immediately notice that the Dirac operator is not a symmetric system in general. Consider for example the half Minkowski spacetime $\mathscr{M}^4 := \mathbb{R} \times [0, \infty)$ endowed with the element line

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2 \,.$$

In this setting the Dirac operators reads as

$$\mathsf{D} = \imath\gamma(e_0)\partial_t + \imath\gamma(e_1)\partial_x + \imath\gamma(e_2)\partial_y + \imath\gamma(e_3)\partial_z \,.$$

Choosing the Dirac representation, $\gamma(e_0), \gamma(e_1), \gamma(e_2)$ and $\gamma(e_3)$, are given by

$$\gamma(e_0) = \begin{pmatrix} \mathrm{Id}_{2\times 2} & 0 \\ 0 & -\mathrm{Id}_{2\times 2} \end{pmatrix} \qquad \gamma(e_j) = \begin{pmatrix} 0 & \sigma_j \\ -\sigma_j & 0 \end{pmatrix} \,,$$

where σ_j, with $j = 1, 2, 3$ are the Pauli matrices. By straightforward computation, we obtain

- $\gamma(e_j)^\dagger = -\gamma(e_j)$, with $j = x, y, z$ violates condition (S);
- $\kappa = 0$ violates condition (P);
- $\sigma_{\mathsf{D}}(dt) = \gamma(e_0)$ is not positive definite, therefore condition (H) is violated.

Nonetheless, it still possible to find a fiberwise invertible endomorphism $Q \in \Gamma(\mathrm{End}(S\mathcal{M}))$ such that locally $\mathfrak{S} := Q \circ \mathsf{D} + \lambda \mathrm{Id}$ is a symmetric positive hyperbolic system, for a suitable $\lambda \in (0, \infty)$, and the Cauchy problem for $\mathfrak{S}$ and D are equivalent.

Lemma 6. *Consider a globally hyperbolic spin manifold $\mathcal{M}$ with boundary $\partial\mathcal{M}$. Let D be the Dirac operator and $\lambda \in \mathbb{R}$. The first order differential operator $\mathfrak{S} : \Gamma(S\mathcal{M}) \to \Gamma(S\mathcal{M})$ defined by*

$$\mathfrak{S} = -\imath\gamma(e_0)\mathsf{D} + \lambda\,\mathrm{Id}$$

is a symmetric hyperbolic system for all $\lambda \in \mathbb{R}$ and its Cauchy problem

$$\begin{cases} \mathfrak{S}\Psi = \mathfrak{f} \in \Gamma_c(S\mathcal{M}) \\ \Psi|_{\Sigma_0} = \mathfrak{h} \in \Gamma_c(S\Sigma_0) \\ \mathfrak{M}\Psi|_{\partial\mathcal{M}} = 0 \end{cases} \tag{2.2}$$

is equivalent to the Cauchy problem for the Dirac operator

$$\begin{cases} \mathsf{D}\psi = f \in \Gamma_c(S\mathcal{M}) \\ \psi|_{\Sigma_0} = h \in \Gamma_c(S\Sigma_0) \\ \mathsf{M}\psi|_{\partial\mathcal{M}} = 0 \end{cases} \tag{2.3}$$

with boundary condition $\mathsf{M}\psi|_{\partial\mathcal{M}} = 0$, where $\mathfrak{M} = -\gamma(e_0)\mathsf{M}$. Moreover, for any compact set $\mathcal{R} \subset \mathcal{M}$, there exists a λ such that $\mathfrak{S}$ is a symmetric positive hyperbolic system.

Proof. Since the $-\imath\gamma(e_0)\mathsf{D}$ is a symmetric hyperbolic system, see e.g. [33, Chapter 3], also $\mathfrak{S}$ satisfies condition (S) in Definition 4. Next we show that the Cauchy problem for $\mathfrak{S}$ and for D are equivalent for any $\lambda \in \mathbb{R}$. Wen set $\Psi = e^{-\lambda t}\psi$ and $\mathfrak{f} = e^{-\lambda t}\gamma(e_0) f$ and obtain

$$\begin{aligned} \mathfrak{f} = \mathfrak{S}\Psi = \mathfrak{S}(e^{-\lambda t}\psi) = (-\imath\gamma(e_0)\mathsf{D} + \lambda \mathrm{Id})(e^{-\lambda t}\psi) \\ = -\,\imath e^{-\lambda t}\gamma(e_0)\mathsf{D}\psi = e^{-\lambda t}\gamma(e_0) f. \end{aligned}$$

Moreover, since $e^{-\lambda t} \neq 0$ for all $t \in \mathbb{R}$, we can conclude

$$\mathfrak{M}\Psi|_{\partial\mathcal{M}} = -\frac{1}{2}e^{-\lambda t}\gamma(e_0)\mathsf{M}\psi|_{\partial\mathcal{M}} = 0 \quad \text{if and only if} \quad \mathsf{M}\psi|_{\partial\mathcal{M}} = 0. \qquad \square$$

Let now $\mathcal{R} \subset \mathcal{M}$ be compact. It remains to check condition (P). The operator defined by κ is a zero order operator give by multiplication by smooth functions. Therefore, for any compact set $\mathcal{R}$, $\kappa|_{\Gamma(S\mathcal{R})}$ is bounded and there exists a suitable λ_j such that $\mathfrak{S}$ is positive definite.

Example 7. Let consider the half Minkowski spacetime $\mathscr{M}^4$ and the Dirac operator

$$\mathsf{D} = \imath\gamma(e_0)\partial_t + \imath\gamma(e_1)\partial_x + \imath\gamma(e_2)\partial_y + \imath\gamma(e_3)\partial_z\,.$$

Here the γ-matrices are given in Remark 5. We can see that the operator

$$\mathfrak{S} = -\imath\gamma(e_0)\mathsf{D} + m\mathrm{Id}$$

is a *symmetric positive hyperbolic system* on account of

$$\big(\gamma(e_0)\gamma(e_i)\big)^\dagger = \gamma(e_i)^\dagger\gamma(e_0)^\dagger = -\gamma(e_0)^\dagger\gamma(e_i)^\dagger = \gamma(e_0)\gamma(e_i) \qquad \text{for all } i = 1,2,3,$$

$$\kappa = \begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix} \mathrm{Id}_{4\times 4} \qquad \text{and} \qquad \sigma_{\mathfrak{S}}(dt) = \mathrm{Id}_{4\times 4}\,.$$

3. Local well-posedness of the Cauchy problem

Let D be the Dirac operator on our globally hyperbolic spin manifold with timelike boundary $\partial\mathcal{M}$. We fix $\mathfrak{h} \in \Gamma_c(S\Sigma_0)$ and $\mathfrak{f} \in \Gamma(S\mathcal{M})$. We denote by $\mathcal{T}$ the *time strip* given by

$$\mathcal{T} := t^{-1}([0,T])$$

where $t\colon \mathcal{M} \to \mathbb{R}$ is the chosen Cauchy time function. Let λ a suitable constant in $\mathbb{R}$ such that the operator $\mathfrak{S}\colon \Gamma(S\mathcal{M}) \to \Gamma(S\mathcal{M})$ defined by

$$\mathfrak{S} = -\imath\gamma(e_0)\mathsf{D} + \lambda\,\mathrm{Id} \tag{3.1}$$

is a symmetric positive hyperbolic system on

$$\mathcal{R}^\wedge := J^-(\mathcal{O}) \cap \mathcal{T}$$

where $\mathcal{O}$ is a compact subset of Σ_T and $J^-(\mathcal{O})$ (resp. $J^+(\mathcal{O})$) denotes the *past set* (resp. *future set*), namely the set of all points that can be reached by past-directed resp. future-directed causal curves emanating from a point in $\mathcal{O}$. This is always possible by Lemma 6 since $\mathcal{R}^\wedge \subset \mathcal{M}$ is compact. For the reason why we choose $\mathcal{R}^\wedge$ as above compare below—especially Theorem 14.

3.1. Uniqueness and finite propagation speed.

In order to show existence and uniqueness of weak solutions for the Dirac Cauchy problem (2.3), we first shall derive the so called *"energy inequality"*. This estimate has a clear physical interpretation, as we shall see in Proposition 10: Any solution can propagate with at most speed of light.

Lemma 8. *Let be $\Psi \in \Gamma(S\mathcal{T})$ satisfying $\Psi|_{\Sigma_0} = 0$ and $\mathfrak{M}\Psi|_{\partial\mathcal{M}} = 0$. Then Ψ satisfies the* energy inequality

$$\|\Psi\|_{L^2(\mathcal{R}^\wedge)} \leq \lambda^{-1}\|\mathfrak{S}\Psi\|_{L^2(\mathcal{R}^\wedge)}\,. \tag{3.2}$$

Proof. By Green identity we obtain

$$(\Psi \,|\, \mathfrak{S}\Psi)_{\mathcal{R}^\wedge} - (\mathfrak{S}^\dagger \Psi \,|\, \Psi)_{\mathcal{R}^\wedge} = (\Psi \,|\, \gamma(e_0)\gamma(\mathtt{n})\Psi)_{\partial\mathcal{R}^\wedge} \tag{3.3}$$

where the $\gamma(\cdot)$ denotes Clifford multiplication and $\mathtt{n}$ is outward normal vector to $\partial\mathcal{R}^\wedge$ and $(\cdot\,|\,\cdot)_{\partial\mathcal{R}^\wedge}$ is the induced L^2-product on $\partial\mathcal{R}^\wedge$. Subtracting $2(\Psi \,|\, \mathfrak{S}\Psi)_{\mathcal{R}^\wedge}$ from the latter equation, we thus obtain

$$\begin{aligned}(\Psi \,|\, \gamma(e_0)\gamma(\mathtt{n})\Psi)_{\partial\mathcal{R}^\wedge} - 2(\Psi \,|\, \mathfrak{S}\Psi)_{\mathcal{R}^\wedge} &= -(\Psi \,|\, \mathfrak{S}\Psi)_{\mathcal{R}^\wedge} - (\Psi \,|\, \mathfrak{S}^\dagger\Psi)_{\mathcal{R}^\wedge} \\ &= -(\Psi \,|\, (\mathfrak{S} + \mathfrak{S}^\dagger)\Psi)_{\mathcal{R}^\wedge} \leq -2(\Psi \,|\, \lambda\Psi)_{\mathcal{R}^\wedge}\,,\end{aligned} \tag{3.4}$$

where in the last inequality we used that $\mathfrak{S}$ is a symmetric positive system. Next, let decompose the boundary $\partial\mathcal{R}^\wedge$ as

$$\partial\mathcal{R}^\wedge = \mathcal{O} \cup \Big(\Sigma_0 \cap J^-(\mathcal{O})\Big) \cup Y\,,$$

where $Y := \partial J^-(\mathcal{O}) \cap \mathcal{T}$, loosely speaking Y is the boundary of the light cone inside the time strip $\mathcal{T}$. The boundary terms on $\Sigma_0 \cap J^-(\mathcal{O})$ vanish by definition of $\Gamma_T(S\mathcal{R}^\wedge)$. Hence we will have non zero boundary contributions only at $\mathcal{O}$ and $Y \subset \partial\mathcal{R}^\wedge$. To deal with these terms, we decompose further the boundary Y in

$$Y = (Y \cap \partial\mathcal{M}) \sqcup \big(Y \setminus (Y \cap \partial\mathcal{M})\big)\,.$$

Notice that if $\partial\mathcal{R}^\wedge \cap \partial\mathcal{M} = \emptyset$ then Y reduces to the boundary of the light cone $J^-(\mathcal{O})$ in the interior of $\mathcal{R}^\wedge$. Imposing the boundary condition $\mathfrak{M}\Psi|_{\partial\mathcal{M}} = 0$, we have by condition (1.2) that

$$(\Psi \,|\, \gamma(e_0)\gamma(\mathtt{n})\Psi)_{\partial\mathcal{R}^\wedge} = 0\,.$$

Therefore, the boundary terms on $Y \cap \partial\mathcal{M}$ vanish. In this way, the Green identity (3.3) reduces to

$$(\Psi \,|\, \mathfrak{S}\Psi)_{\mathcal{R}^\wedge} - (\mathfrak{S}^\dagger \Psi \,|\, \Psi)_{\mathcal{R}^\wedge} = (\Psi \,|\, \gamma(e_0)\gamma(\mathtt{n})\Psi)_{\mathcal{O}} + (\Psi \,|\, \gamma(e_0)\gamma(\mathtt{n})\Psi)_{Y\setminus(Y\cap\partial\mathcal{M})}.$$

Let us remark that the right hand side of the latter equation is non negative definite. Indeed, since $\mathcal{O}$ is a spacelike hypersurface, $(.\,|\,\gamma(e_0)\gamma(\mathtt{n}).)_{\mathcal{O}}$ is an inner product by [17, Prop. 1.1] and, thus,

$$(\Psi \,|\, \gamma(e_0)\gamma(\mathtt{n})\Psi)_{\mathcal{O}} > 0\,.$$

By continuity, also the contribution on $Y \setminus (Y \cap \partial\mathcal{M})$ is still positive semidefinite since $Y \setminus (Y \cap \partial\mathcal{M})$ is a lightlike hypersurface. Hence, plugging this consideration into Inequality (3.4) we obtain

$$2(\Psi \,|\, \lambda\Psi)_{\mathcal{R}^\wedge} \leq 2(\Psi \,|\, \mathfrak{S}\Psi)_{\mathcal{R}^\wedge}.$$

Thus, by Hölder inequality we have for all $\Psi \in \Gamma_T(S\mathcal{R}^\wedge)$

$$\|\Psi\|_{L^2(\mathcal{R}^\wedge)} \leq \lambda^{-1}\|\mathfrak{S}\Psi\|_{L^2(\mathcal{R}^\wedge)}.$$

□

Remark 9. Notice that choosing zero initial data is not a loss of generalities. In fact, for any Cauchy problem with any nonzero initial data $\mathfrak{h} \in \Gamma_c(S\Sigma_0)$ there exists an equivalent Cauchy problem with zero initial data, namely

$$\begin{cases}\mathfrak{S}\Psi = \mathfrak{f} \\ \Psi|_{\Sigma_0} = \mathfrak{h} \\ \mathfrak{M}\Psi|_{\partial\mathcal{M}} = 0\end{cases} \iff \begin{cases}\mathfrak{S}\widetilde{\Psi} = \widetilde{\mathfrak{f}} \\ \widetilde{\Psi}|_{\Sigma_0} = 0 \\ \mathfrak{M}\widetilde{\Psi}|_{\partial\mathcal{M}} = 0\end{cases}$$

for any $\mathfrak{f}, \widetilde{\mathfrak{f}} \in \Gamma_c(S\mathcal{M})$. Here $\widetilde{\Psi}(t,\vec{x}) := \Psi(t,\vec{x}) + \mathfrak{h}(\vec{x})$ and $\widetilde{\mathfrak{f}}(t,\widetilde{x}) := \mathfrak{f}(t,\widetilde{x}) + \lambda\mathfrak{h}(\widetilde{x}) + \sum_{j=1}^{n} \gamma(e_0)\gamma(e_j)\nabla_{x^j}\mathfrak{h}$. Note that $\mathfrak{f} \in \Gamma_{cc}(S\mathcal{M})$ if and only if $f \in \Gamma_{cc}(S\mathcal{M})$.

We are now ready to prove that if there exists a solution to the Cauchy problem (2.2), then it propagates with at most speed of light.

Proposition 10 (Finite speed of propagation)**.** *Any solution to the Dirac Cauchy problem* (2.3) *propagates with at most speed of light.*

Proof. Consider any point p outside the region

$$\mathcal{V} := \Big(J^+\big(\text{supp }\,\mathfrak{f} \cap \mathcal{T}\big) \cup J^+(\text{supp }\mathfrak{h})\Big) \cap \mathcal{T}$$

as Figure 1. Then there exists a λ such that $\mathfrak{S}$ is a symmetric positive hyperbolic

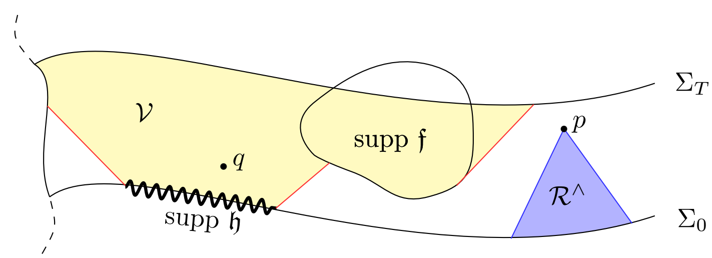

FIGURE 1. Finite propagation of speed.

system on $\mathcal{R}^\wedge = \mathcal{T} \cap J^-(p)$. Since $p \in \mathcal{R}^\wedge$ cannot be reached by a future-directed causal curve starting in any point $q \in \mathcal{V}$ we have $\mathfrak{S}\Psi = 0$ and $\Psi|_{\mathcal{T}\setminus\mathcal{V}} = 0$. By Lemma 8, Ψ vanishes in $\mathcal{R}^\wedge$. Hence, Ψ vanishes outside $J^+(\mathcal{V})$.
The finite propagation of speed of a solution of the Dirac Cauchy problem (2.3) then follows by Lemma 6. □

Remark 11. As a byproduct of Proposition 10, any solution Ψ of the Cauchy problem (2.2) in the time strip $\mathcal{T}$ is identically zero outside the set $\mathcal{T} \cap \mathcal{V}$ with

$$\mathcal{V} := \Big(J^+\big(\text{supp }\,\mathfrak{f} \cap \mathcal{T}\big) \cup J^+(\text{supp }\mathfrak{h})\Big) \cap \mathcal{T}\,.$$

Next, we prove that if a solution exists, then it is unique.

Proposition 12 (Uniqueness)**.** *Suppose there exist $\Psi, \Phi \in \Gamma(S\mathcal{T})$ satisfying the same Cauchy problem* (2.2)*. Then $\Psi = \Phi$. In particular, this also gives uniqueness of solution for the Dirac Cauchy problem* (2.3)*.*

Proof. Since Ψ and Φ satisfy the same initial-boundary value problem (2.2), then $\Psi - \Phi \in \Gamma(S\mathcal{T})$ is a solution of

$$\begin{cases} \mathfrak{S}(\Psi - \Phi) = 0 \\ (\Psi - \Phi)|_{\Sigma_0} = 0 \\ \mathfrak{M}(\Psi - \Phi)|_{\partial\mathcal{M}} = 0 \end{cases}$$

By Proposition (10), the supports of Ψ and Φ are contained in $\mathcal{R}^\wedge$ for $\mathcal{O} := \mathcal{V} \cap \Sigma_T$. Therefore, we can use Lemma 8 to conclude that $\Psi - \Phi$ is zero.

The uniqueness of the Dirac Cauchy problem (2.3) then follows by Lemma 6. □

We conclude this section by deriving an energy inequality for the formal adjoint of $\mathfrak{S}$ analogous to (3.2). Here we restrict to the set $\mathcal{R}^\vee$ defined by

$$\mathcal{R}^\vee := \mathcal{T} \cap J^+(\mathcal{O}') \tag{3.5}$$

for a compact domain $\mathcal{O}' \subset \Sigma_0$ with smooth boundary. Using the same arguments as in the proof of Lemma 8, we conclude:

Lemma 13. *Let denote with $\mathfrak{S}^\dagger$ and $\mathfrak{M}^\dagger$ the formal adjoint of $\mathfrak{S}$ and $\mathfrak{M}$ respectively. Moreover, let be $\Phi \in \Gamma(S\mathcal{T})$ satisfying $\Phi|_{\Sigma_T} = 0$ and $\mathfrak{M}^\dagger\Phi|_{\partial\mathcal{M}} = 0$. Then Φ satisfies the inequality*

$$\|\Phi\|_{L^2(\mathcal{R}^\vee)} \leq C\|\mathfrak{S}^\dagger\Phi\|_{L^2(\mathcal{R}^\vee)}\,. \tag{3.6}$$

3.2. **Existence of weak solutions in a time strip.** With the help of the Energy inequality (3.2) we shall prove the existence of a *weak solution* for the mixed initial-boundary value problem (2.2) for $\mathfrak{S}$ as in (3.1) (still for fixed $\mathfrak{h} \in \Gamma_c(S\Sigma_0)$ and $\mathfrak{f} \in \Gamma(S\mathcal{M})$). To this end, let λ such that $\mathfrak{S}^\dagger$ is a symmetric positive hyperbolic system on $\mathcal{R}^\vee$ for $\mathcal{O}' := J^-(\mathcal{V} \cap \Sigma_T) \cap \Sigma_0$ with

$$\mathcal{V} := \Big(J^+\big(\text{supp } \mathfrak{f} \cap \mathcal{T}\big) \cup J^+(\text{supp } \mathfrak{h})\Big) \cap \mathcal{T}\,. \tag{3.7}$$

Moreover, let us endow $\Gamma_c(S\mathcal{T})$ with L^2-scalar product

$$(\cdot\,|\,\cdot)_{L^2(\mathcal{T})} := \int_{\mathcal{T}} \langle\cdot\,|\,\cdot\rangle \mathrm{Vol}_{\mathcal{T}}\,,$$

where $\mathrm{Vol}_{\mathcal{T}}$ denotes the volume element and $\langle\cdot\,|\,\cdot\rangle$ is the canonical fiberwise scalar product in the spinor bundle. We denote by

$$\mathscr{H} := \overline{\big(\Gamma_c(S\mathcal{T}), (.\,|\,.)_{L^2(\mathcal{T})}\big)}^{(.\,|\,.)_{L^2(\mathcal{T})}}$$

the corresponding completion. We denote by $\|\cdot\|_{L^2(\mathcal{T})}$ the norm induced by $(.\,|\,.)_{L^2(\mathcal{T})}$.

Theorem 14 (Weak existence)**.** *There exists a unique* weak solution $\Psi \in \mathscr{H}$ *to the Cauchy problem* (2.2)*, namely the relation*

$$(\Phi\,|\,\mathfrak{f})_{L^2(\mathcal{T})} = (\mathfrak{S}^\dagger\Phi\,|\,\Psi)_{L^2(\mathcal{T})} \tag{3.8}$$

holds for all $\Phi \in \Gamma_c(S\mathcal{T})$ satisfying $\mathfrak{M}^\dagger\Phi|_{\partial\mathcal{M}} = 0$.

Proof. In view of Proposition 10 and Proposition 12, if there exists a solution Ψ of the Cauchy problem (2.2) then it is unique and it vanishes outside $\mathcal{V}$ defined by 3.7. As a consequence, the scalar product (3.8) is potentially non zero only in $\mathcal{R}^\vee \subset \mathcal{T}$ of the form (3.5) for $\mathcal{O}' := J^-(\mathcal{V} \cap \Sigma_T) \cap \Sigma_0$. Therefore, the scalar product (3.8) can be rewritten as

$$(\Phi\,|\,\mathfrak{f})_{L^2(\mathcal{R}^\vee)} = (\mathfrak{S}^\dagger\Phi\,|\,\Psi)_{L^2(\mathcal{R}^\vee)}.$$

On account of Lemma 13 and using arguments similar to Proposition 12, we can notice that the kernel of the operator $\mathfrak{S}^\dagger$ is trivial. Therefore, $\mathfrak{S}^\dagger$ defines a bijection between $\Gamma(S\mathcal{T})$ and its image CoD. Let now $\ell\colon CoD \to \mathbb{C}$ be the linear functional defined by

$$\ell(\Theta) = (\Phi\,|\,\mathfrak{f})_{L^2(\mathcal{R}^\vee)}$$

where Φ satisfies $\mathfrak{S}^\dagger\Phi = \Theta$. By the energy inequality (3.6), ℓ is bounded:

$$\begin{aligned}\ell(\Theta) = (\Phi \,|\, \mathfrak{f})_{L^2(\mathcal{R}^\vee)} &\leq \|\mathfrak{f}\|_{L^2(\mathcal{R}^\vee)} \|\Phi\|_{L^2(\mathcal{R}^\vee)} \\ &\leq \lambda^{-1}\|\mathfrak{f}\|_{L^2(\mathcal{R}^\vee)}\|\mathfrak{S}^\dagger\Phi\|_{L^2(\mathcal{R}^\vee)} = \lambda^{-1}\|\mathfrak{f}\|_{L^2(\mathcal{R}^\vee)}\|\Theta\|_{L^2(\mathcal{R}^\vee)},\end{aligned}$$

where in the first inequality we used Cauchy-Schwartz inequality. By Hahn-Banach theorem, ℓ can be extended to a continuous functional defined on the completion of CoD with respect to scalar product $\prec \cdot \,|\cdot \succ := (\mathfrak{S}^\dagger \cdot \,|\, \mathfrak{S}^\dagger \cdot)_{L^2(\mathcal{R}^\vee)}$. It is easy to notice that $\mathcal{H} := \overline{(CoD, \prec \cdot\,|\cdot \succ)}^{\prec\cdot\,|\cdot\succ} \subset \mathscr{H}$: Every Cauchy sequence which converges in the norm induced by $\prec \cdot|\cdot \succ$ converges also in the L^2-norm. Finally, by Riesz representation theorem, there exists a unique element $\Psi \in \mathcal{H}$ such that

$$\ell(\Theta) = (\Theta \,|\, \Psi)_{L^2(\mathcal{R}^\vee)}\,.$$

Thus, we obtain

$$(\Phi \,|\, \mathfrak{f})_{L^2(\mathcal{R}^\vee)} = \ell(\Theta) = (\Theta \,|\, \Psi)_{L^2(\mathcal{R}^\vee)} = (\mathfrak{S}^\dagger\Phi \,|\, \Psi)_{L^2(\mathcal{R}^\vee)} = (\Phi \,|\, \mathfrak{S}\Psi)_{L^2(\mathcal{R}^\vee)}$$

where we used in the last step that the boundary terms in the Green identity (3.3) only occur on $\partial\mathcal{M} \cap \mathcal{T}$ and vanish due to (1.2). This concludes our proof. □

3.3. **Differentiability of the solutions.** In the last section we proved that the Cauchy problem (2.2) on a finite time strip admits a unique weak solution. Next we want to show that the solution is a strong solution and in particular it is smooth. Since this is a local question, we want to use the theory for hyperbolic systems on subsets of $\mathbb{R}^{n+1}$—in particular [31, Section 1] that a weak solution is a (semi-)strong solution and the regularity estimates for strong solution in [34, Theorem 3.1]. Since the definition of strong solution in the sense Lax–Phillips does not coincide with the one given by Rauch–Massey, we will denote it as semi-strong solution.

Definition 15. Let $U \subset \mathcal{M}$ be a compact subset in $\mathcal{M}$ with timelike boundary $\partial\mathcal{M}$. We say that $\Psi \in \mathscr{H}$ is a *semi-strong solution* of the initial-boundary value problem (2.2) if there exists a sequence of sections $\Psi_k \in W^{1,2}(\Gamma(SU))$ such that $\mathfrak{M}\Psi_k = 0$ on $\partial\mathcal{M} \cap U$ and

$$\|\Psi_k - \Psi\|_{L^2(U)} \xrightarrow{k\to\infty} 0 \quad \text{and} \quad \|\mathfrak{S}\Psi_k - \mathfrak{f}\|_{L^2(U)} \xrightarrow{k\to\infty} 0.$$

The solution is called *strong* if additionally the sequence Ψ_k can chosen to be smooth.

We concentrate on points in the boundary $p \in \partial\mathcal{M}$ and firstly define a convenient chart as follows, compare Figure 2. Let Σ_p be the Cauchy surface of $\mathcal{M}$ to which p belongs to. For $q \in \partial\mathcal{M}$ let $\widehat{\Sigma}_p := \Sigma_p \cap \partial\mathcal{M}$ be the corresponding Cauchy surface in the boundary. Let $\varrho\colon [0,\varepsilon] \to \partial\mathcal{M}$ be the timelike geodesic in $\partial\mathcal{M}$ starting at p with velocity $\mathbf{v} \in T_p\partial\mathcal{M}$ where $\mathbf{v}$ is a normalized, future-directed, timelike vector perpendicular to $\widehat{\Sigma}_p$. Let $\widehat{B}_\varepsilon(\varrho(t))$ be the ε-ball in $\hat{\Sigma}_{\varrho(t)}$ around $\varrho(t)$. On this balls we choose geodesic normal coordinates $\widehat{\kappa}_t\colon B_\varepsilon^{n-1}(0) \to \widehat{B}_\varepsilon(\varrho(t))$. Moreover, inside each $\Sigma_{\varrho(t)}$ we choose Fermi coordinates with base $\widehat{B}_\varepsilon(\varrho(t))$ and thus get a chart in $\Sigma_{\varrho(t)}$ around $\varrho(t)$ as

$$\begin{aligned}\widetilde{\kappa}_t\colon B_\varepsilon^{n-1}(0) \times [0,\varepsilon] \subset \mathbb{R}^n &\to U_\varepsilon(\widehat{B}_\varepsilon(\varrho(t))) := \{q \in \Sigma_{\varrho(t)} \mid \mathrm{dist}_{\Sigma_{\varrho(t)}}(q, \widehat{B}_\varepsilon(\varrho(t)) \leq \varepsilon\} \\ (y,z) &\mapsto \exp^{\perp,\Sigma_{\varrho(t)}}_{\widehat{\kappa}_t(y)}(z)\end{aligned}$$

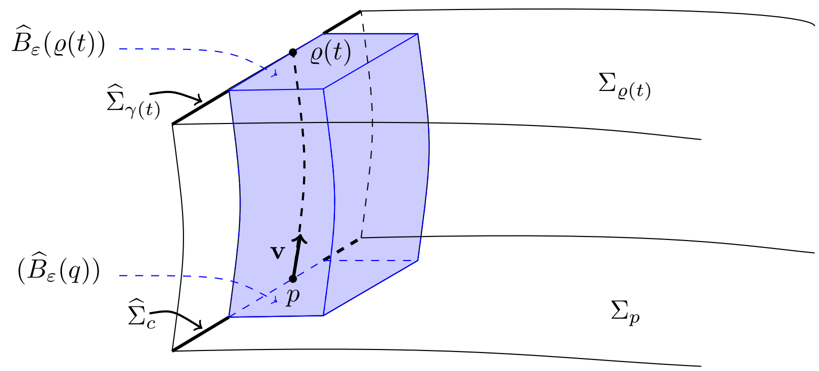

FIGURE 2. Fermi coordinates on each Cauchy surface.

where $\exp^{\perp,\Sigma_{\varrho(t)}}_{\widehat{\kappa}_t(y)}(z)$ is the exponential map in $\Sigma_{\varrho(t)}$ starting at $\widehat{\kappa}_t(y)$ with velocity perpendicular to $\widehat{\Sigma}_{\varrho(t)} = \partial\Sigma_{\varrho(t)}$ pointing in the interior and magnitude z.
Putting all this together we obtain a chart

$$\kappa_p \colon [0,\varepsilon] \times B^{n-1}_\varepsilon(0) \times [0,\varepsilon] \subset \mathbb{R}^{n+1} \to U_p := \bigcup_{t\in[0,\varepsilon]} U_\varepsilon(\widehat{B}_\varepsilon(\varrho(t))) \subset \mathcal{M}$$
$$(t,y,\bar{z}) \mapsto \widetilde{\kappa}_t(y,\bar{z}).$$

Note that sections of the spinor bundle SU_p are now simply vector-valued functions $U_p \to \mathbb{C}^N$ where N is the rank of the spinor bundle.
In these coordinates our Cauchy problem will take the form as in [31, 34]. To see this, let us first consider the model case of 'half' of the Minkowski space.

Example 16. Let $\widetilde{\mathscr{M}^n}$ be the Minkowski space with coordinates $x = (x^0,\dots,x^n)$. We set $t := x^0$, $y = (x^1,\dots,x^{n-1})$ and $z := x^n$. For $|a| < 1$ the hypersurface $\mathcal{N}_a := \{z = at\}$ is timelike and $\mathscr{M}^n_a = \{z \geq at\}$ is a globally hyperbolic manifold with a timelike boundary. We set $\widetilde{z} := z - at$ and use $(t,y,\widetilde{z})$ as new coordinates on $\mathscr{M}^n_a$. Then, together with $\gamma(e_0) = \gamma(e_0)^{-1}$, we have

$$\begin{aligned}
-\imath\gamma(e_0)\mathsf{D} &= \partial_t + \sum_{j=1}^{n-1}\gamma(e_0)\gamma(e_j)\partial_{x^j} + \gamma(e_0)\gamma(e_n)\partial_z \\
&= \big(1 - a\gamma(e_0)\gamma(e_n)\big)\partial_t + \sum_{j=1}^{n-1}\gamma(e_0)\gamma(e_j)\partial_{x^j} + \gamma(e_0)\gamma(e_n)\partial_{\widetilde{z}}.
\end{aligned}$$

Since $a < 1$, the coefficient in front of ∂_t is invertible. Thus, $\widehat{\mathfrak{S}} := (1-a\gamma(e_0)\gamma(e_n))^{-1}\mathfrak{S}$ with $\mathfrak{S} = -\imath\gamma(e_0)\mathsf{D} + m\,\mathrm{Id}$, as in Example 7, is just given by

$$\widehat{\mathfrak{S}} = \partial_t + \underbrace{\sum_{j=1}^{n-1} A_j(x)\partial_{x^j} + A_{\widetilde{z}}(x)\partial_{\widetilde{z}} + B(x)}_{=:-\mathfrak{G}}\,. \tag{3.9}$$

Since $A_{\widetilde{z}}(x) = \big(1 - a\gamma(e_0)\gamma(e_n)\big)^{-1}\gamma(e_0)\gamma(e_n)$ is in particular nonsingular on $\partial\mathscr{M}^n_a$ and since $\ker\mathfrak{M}|_q$ varies smoothly with $q \in \partial\mathscr{M}^n_a$, after restricting to some cube in $\mathscr{M}^n_a$ we are exactly in the situation considered in [31, 34].

Corollary 17. *For the Dirac operator D on a globally hyperbolic spin manifold with timelike boundary $\partial\mathcal{M}$ and $p \in \partial\mathcal{M}$, there is a sufficiently small $\varepsilon > 0$ such that in the coordinates κ_p from above there is an invertible operator $\mathfrak{E}\colon \Gamma(U_p, \mathbb{C}^N) \to \Gamma(U_p, \mathbb{C}^N)$ such that $\mathfrak{E}\mathfrak{S}$ has the form (3.9) with $A_{\bar{z}}$ nonsingular on the boundary. In particular, by Theorem 14 we have a weak solution to the Cauchy problem*

$$\begin{cases} \widehat{\mathfrak{S}} := \mathfrak{E}\mathfrak{S}\Psi & = \mathfrak{f} \in \Gamma_c(SU_p) \\ \Psi|_{V_p} & = \mathfrak{h} \in \Gamma_c(SV_p) \\ \mathfrak{M}\Psi|_{\partial\mathcal{M}} & = 0 \end{cases} \tag{3.10}$$

on U_p where $\mathfrak{S}$ is as in Theorem 14 and $V_p := U_p \cap t^{-1}(0)$.

Proof. By the choice of the coordinates the Dirac operator will look in p exactly as for the Minkowski space as computed in Example 16. Note that the role of $\mathcal{N}_a$ is taken by the tangent plane of $\partial\mathcal{M} \hookrightarrow \mathcal{M}$ in p. Since everything is continuous we can find a sufficiently small $\varepsilon > 0$ such that there is an invertible linear map $\mathfrak{E}\colon \Gamma(U_p, \mathbb{C}^N) \to \Gamma(U_p, \mathbb{C}^N)$ with $\mathfrak{E}|_{S_p\mathcal{M}} = -\imath\big(1 - a\gamma(e_0)\gamma(e_n)\big)^{-1}\gamma(e_0)$ such that $\mathfrak{E}\mathfrak{S}$ has the required form.

Moreover, we obtain a weak solution as required by taking the weak solution of Theorem 14 where the right handside is given by $(\mathfrak{E}^{-1}\mathfrak{f}, \mathfrak{h}, 0)$. □

Lemma 18 (Local strong solution)**.** *Then the weak solution Ψ of the Cauchy problem (3.10) is locally a strong solution.*

Proof. The last Corollary tells us that we can apply [31, Section 2] in order to obtain the existence of a semi-strong solution, i.e. there is a sequence of continuous sections $\Phi_k \in W^{1,2}(U_p, \mathbb{C}^N)$ with $\mathfrak{M}\Phi_k|_{\partial\mathcal{M}\cap U_p} = 0$ and $\|\Phi_k - \Psi\|_{L^2(U_p)} \to 0$ and $\|\widehat{\mathfrak{S}}\Phi_k - \mathfrak{f}\|_{L^2(U_p)} \to 0$. It remains to argue, that we can approximate the Φ_k by smooth Ψ_k still fulfilling the boundary condition and the convergences from above.

This can be achieved using the standard theory of Sobolev spaces. We refer to [18] for more details. First, choose $u_i \in \Gamma(U_p, \mathbb{C}^N)$, $i = 1, \ldots, r$, such that for each $q \in \partial\mathcal{M} \cap U_p$ they form a basis of $\ker \mathfrak{M}|_q$ and are linearly independent in all $q \in U_p$. Since $\mathcal{M}$ depends smoothly on the base point and has constant rank this is always possible. Choose $u_j \in \Gamma(U_p, \mathbb{C}^N)$, $j = r+1, \ldots, N$ such that $u_1(q), \ldots, u_N(q)$ is a basis of $\mathbb{C}^N$ at each $q \in U_p$. A section $\Phi \in W^{1,2}(U_p, \mathbb{C}^N)$ can now be expressed as $\Phi = \sum_{i=1}^N a_i u_i$ for $a_i\colon U_p \to \mathbb{C}$. We denote by Φ^+ the part of Φ spanned by u_1 to u_r and set $\Phi^- := \Phi - \Phi^+$. Using the a_i as the new coordinates, we have $\Psi_k^+ \in W^{1,2}(U_p, \mathbb{C}^r)$ and $\Psi_k^- \in W^{1,2}(U_p, \mathbb{C}^{N-r})$. Thus, there is a sequence $\Psi_{k,j}^+ \in \Gamma(U_p, \mathbb{C}^r)$ that converges to Ψ_k^+ in $W^{1,2}$ and analogously a smooth sequence $\Psi_{k,j}^-$ converging to Ψ_k^- in $W^{1,2}$. Moreover, by definition $\mathtt{tr}\,\Phi_k^+ = 0$, where $\mathtt{tr}$ is the trace map $W^{1,2}(U_p, \mathbb{C}^r) \to L^2(U_p \cap \partial\mathcal{M}, \mathbb{C}^r)$. Thus, $\Phi_{k,j}^+$ can be chosen to be zero on $U_p \cap \partial\mathcal{M}$. Thus, $\Phi_{k,j} = \Phi_{k,j}^+ + \Phi_{k,j}^-$, where we use the embeddings $\mathbb{C}^r \hookrightarrow \mathbb{C}^N$ and $\mathbb{C}^r \hookrightarrow \mathbb{C}^{N-r}$ from above, are smooth sections fulfilling $\mathfrak{M}\Phi_{k,j}|_{U_p\cap\partial\mathcal{M}} = 0$. Choosing a diagonal sequence we obtain smooth Ψ_k approximating Ψ as in Definition 15. □

Next we want to see whether the strong solution is actually smooth. Just assuming $\mathfrak{f} \in \Gamma_c(S\mathcal{M})$ and $\mathfrak{h} \in \Gamma_c(S\Sigma_0)$ (and not as in Theorem 1 compactly supported in the interior) is not sufficient to guarantee that the solution of the Cauchy problem (2.2) is smooth.

Example 19. Let $\mathscr{M}_a^n$ be the half Minkowski spacetime as described in Example 16 and consider the Cauchy problem (3.10). Assume that $\mathfrak{M}$ does not depend on t, that is (e.g.) true for MIT boundary conditions. In local coordinates the Cauchy problem reads as

$$\begin{cases} \mathfrak{S}\Psi = (\partial_t - \mathfrak{G})\Psi = \mathfrak{f} \\ \Psi|_{t=0} = \mathfrak{h} \\ \mathfrak{M}\Psi|_{\tilde{z}=0} = 0. \end{cases}$$

Suppose that Ψ is k-differentiable. Then it satisfies

$$0 = \partial_t^k\big(\mathfrak{M}\Psi|_{\tilde{z}=0}\big)|_{t=0} = \big(\mathfrak{M}\partial_t^k\Psi|_{\tilde{z}=0}\big)|_{t=0} = \big(\mathfrak{M}\mathfrak{G}^k\Psi(t)|_{\tilde{z}=0}\big)|_{t=0} = \mathfrak{M}\big(\mathfrak{G}^k\mathfrak{h}\big)|_{\tilde{z}=0}\,.$$

Therefore, any initial data have to satisfy a compatibility condition.

Remark 20. Let us remark that in the previous example we massively used that $\mathfrak{M}$ and $\mathfrak{G}$ does not depend on t. In the more general case of $\mathfrak{G}$ for a Dirac operator on a globally hyperbolic manifold in the coordinates on U_p defined on page 10 this is in general not the case. A general compatibility condition on $\mathfrak{f}$ and $\mathfrak{h}$ can be obtained by setting

$$\mathfrak{h}_k = \sum_{j=0}^{k-1} \frac{(k-1)!}{j!(k-1-j)!}\big(\partial_t^j\mathfrak{G}\big)|_{V_p}\mathfrak{h}_{k-1-j} + \partial_t^k\mathfrak{f}|_{V_p}$$

with $\mathfrak{h}_0 = \mathfrak{h}$, $V_p = U_p \cap \Sigma_0$ and imposing that the data $\mathfrak{h} \in \Gamma_c(S\Sigma_0)$ and $\mathfrak{f} \in \Gamma(S\mathcal{M})$ satisfy

$$\sum_{j=1}^{k} \frac{k!}{(k-j)!}(\partial_t^j\mathfrak{M})|_{V_p}\mathfrak{h}_{j-1} = 0$$

for all $k \geq 1$. Here $\mathfrak{G}$ has the form (3.9). Translating this back for our Dirac Cauchy problem (2.3) in the Hamiltonian form

$$(\partial_t - \mathsf{H})\psi = \gamma(e_0)f$$

the compatibility condition for $h \in \Gamma_c(S\Sigma_0)$ and $\mathfrak{f} \in \Gamma(S\mathcal{M})$ reduces to

$$\sum_{j=1}^{k} \frac{(k)!}{j!(k-j)!}\Big(\partial_t^j\mathsf{M}\Big)\Big|_{\partial\Sigma_0} h_{k-1} = 0$$

for all $k \geq 1$ where

$$h_k = \sum_{j=0}^{k-1} \frac{(k-1)!}{j!(k-1-j)!}(\partial_t^j\mathsf{H})|_{\partial\Sigma_0}\, h_{k-1-j} + \partial_t^p\big(\gamma(e_0)f\big)|_{\partial\Sigma_0}$$

with $h_0 = h$.

The claim in Remark 20 follows by localizing a solution in any set U_p defined as above and then applying [34, Theorem 3.1].

Corollary 21 (Local smooth solution)**.** *Let U_p as above. Then the strong solution for the Cauchy problem (2.2) is smooth if and only if $\mathfrak{h} \in \Gamma_c(U_p, \mathbb{C}^N)$ and $\mathfrak{f} \in \Gamma(U_p, \mathbb{C}^N)$ satisfy*

$$\sum_{j=1}^{k} \frac{k!}{(k-j)!}(\partial_t^j\mathfrak{M})|_{\partial\Sigma_0}\,\mathfrak{h}_{k-1} = 0\,. \tag{3.11}$$

Remark 22. If we choose an initial data $\mathfrak{h}$ resp. $\mathfrak{f}$ with compact support in the interior of Σ_0 resp. $\mathcal{M}$, the compatibility condition is automatically satisfied. Actually for $\mathfrak{f}$ it is enough to be zero in a neighborhood of $\partial\Sigma_0 \subset \mathcal{M}$.

Corollary 23. *The weak solution of the Cauchy problem* (3.10) *for* $\mathfrak{f} \in \Gamma_{cc}(S\mathcal{M})$ *and* $\mathfrak{h} \in \Gamma_{cc}(S\Sigma_0)$ *in the time strip* $\mathcal{T}$ *provided by Theorem 14 is smooth. In particular, there is a smooth solution of the Dirac Cauchy problem* (1.1) *in* $\mathcal{T}$.

Proof. First let $p \in \partial\mathcal{M} \cap \Sigma_{\hat{t}}$ for some $\hat{t} \in [0, T]$. Let $\varrho \colon [0, \hat{t}] \to \partial M$ be a timelike curve with $\varrho(0) \in \Sigma_0$ and $p = \varrho(\hat{t})$. We fix $\varepsilon > 0$ such that we have Fermi coordinates on a 'cube' $U_{\varrho(t)}$ around $\varrho(t)$ as in Section 3.3 for all $t \in [0, \hat{t}]$ and such that Corollary 17 holds for those cubes. This is always possible since the image of ϱ is compact and everything depends smoothly on the basepoints.

For $U_{\varrho(t)}$ we know that the compatibility condition (3.11) is fulfilled by assumption. Thus, Corollary 21 tells us that the weak solution Ψ is smooth in $U_{\varrho(0)}$ and that for every $a \in [0, \varepsilon]$ the function $\mathfrak{h}_a := \Psi|_{U_{\varrho(0)} \cap \Sigma_a}$ as new initial data $\mathfrak{h}$ together with the original $\mathfrak{f}$ still fulfill the compatibility condition. Moreover, $\Psi|_{U_{\varrho(a)}}$ is still a weak solution to the initial data $(\mathfrak{h}_a, \mathfrak{f})$ on $U_{\varrho(a)}$. Thus, we can again use Corollary 21 where Σ_a now takes the role of Σ_0. Iterating this procedure, we obtain smoothness on all $U_{\varrho(t)}$ for $t \in [0, \hat{t}]$, i.e. in particular in p.

For $p \in \mathcal{M} \setminus \partial\mathcal{M}$ we choose a timelike curve $\varrho \colon [0, \hat{t}] \to \mathcal{M} \setminus \partial\mathcal{M}$ with $\varrho(0) \in \Sigma_0$ and $p = \varrho(T)$ and proceed as before. It is even easier since we can just use geodesic normal coordinates in the Cauchy surfaces around each $\rho(t)$.

The existence of smooth solutions to the Dirac Cauchy problem (2.3) then follows by Lemma 6. □

Remark 24. In view of Remark 2, we want to comment on the assumption on the boundary condition that we have used up to here. The energy inequalities (3.2) and (3.6) need that $\mathfrak{M}\Psi|_{\partial\mathcal{M}} = 0$ implies

$$\langle \Psi \,|\, \gamma(e_0)\gamma(\mathtt{n})\Psi \rangle_q = 0$$

for all $q \in \partial\mathcal{M}$. Moreover, in order to apply [31] in Lemma 18 and [34] in Corollary 21, we additionally use that $\ker \mathfrak{M}|_q$ is nonempty and varies smoothly with $q \in \partial\mathcal{M}$.

The properties collected above are valid for the MIT bag boundary condition $\mathsf{M} := (\gamma(\mathtt{n}) - \imath)$ (and analogously for $\mathsf{M} := (\gamma(\mathtt{n}) + \imath)$). Indeed, on account of

$$\begin{aligned}\langle \Psi \,|\, \gamma(e_0)\gamma(\mathtt{n})\Psi \rangle_q = \langle \gamma(e_0)\Psi \,|\, \gamma(\mathtt{n})\Psi \rangle_q = \langle -\gamma(\mathtt{n})\gamma(e_0)\Psi \,|\, \Psi \rangle_q = \\ = \langle \gamma(e_0)\gamma(\mathtt{n})\Psi \,|\, \Psi \rangle_q = \overline{\langle \Psi \,|\, \gamma(e_0)\gamma(\mathtt{n})\Psi \rangle_q}\end{aligned}$$

we obtain $\langle \Psi \,|\, \gamma(e_0)\gamma(\mathtt{n})\Psi \rangle_q \in \mathbb{R}$. By using MIT boundary conditions, we have

$$\begin{aligned}\langle \Psi \,|\, \gamma(e_0)\gamma(n)\Psi \rangle_q = \langle \Psi \,|\, \gamma(e_0)\imath\Psi \rangle_q = \langle \gamma(e_0)\Psi \,|\, \imath\Psi \rangle_q = \\ = \langle -\gamma(e_0)\imath\Psi \,|\, \Psi \rangle_q = \langle -\gamma(e_0)\gamma(\mathtt{n})\Psi \,|\, \Psi \rangle_q = -\overline{\langle \Psi \,|\, \gamma(e_0)\gamma(n)\Psi \rangle_q}\,,\end{aligned}$$

which implies $\langle \Psi \,|\, \gamma(e_0)\gamma(\mathtt{n})\Psi \rangle_q = 0$.

Another example is the chirality operator $\mathsf{M} := (\mathrm{Id} - \gamma(n)\mathcal{G})$ where $\mathcal{G}$ is the restriction to $\partial\mathcal{M}$ of an endomorphism-field of $S\mathcal{M}$ which is involutive, unitary, parallel and anti-commuting with the Clifford multiplication on $\mathcal{M}$, [28, Section 1.5].

4. Global well-posedness of Cauchy problem

Up to now we obtained a weak solution in a time strip and showed that this actually smooth if the initial data are compactly supported in the interior (or more generally fulfill the compatibility condition (3.11), compare Remark 22). We can finally put everything together to obtain global well-posedness of the Cauchy problem (1.1).

Proof of Theorem 1. Fix $h \in \Gamma_{cc}(S\Sigma_0)$. By Corollary 21, for any $T \in [0,\infty)$ there exists a smooth solution ψ_T to the Dirac Cauchy problem (1.1) in the time strip $\mathcal{T}_T := t^{-1}([0,T])$. Now consider $T_1, T_2 \in [0,\infty)$ with $T_2 > T_1$. By uniqueness of solution, see Proposition 10, we have $\psi_{T_2}|_{\mathcal{T}_{T_1}} = \psi_{T_1}$. Hence, we can glue everything together to obtain a smooth solution for all $\mathcal{T} \geq 0$. A similar arguments holds for negative time.
Since $h \in \Gamma_{cc}(S\Sigma_0)$, $f \in \Gamma_{cc}(S\mathcal{M})$ it follows by the finite propagation of speed, see Proposition 10, that the solution is spacelike compact. For the continuous dependency on the initial data, see the remark below. □

We are now in the position to discuss the stability of the Cauchy problem. Since the proof is independent on the presence of the boundary and it does rely mostly on functional analytic techniques, we shall omit it and we refer to [1, Section 5] for further details.

Proposition 25. *Consider a globally hyperbolic spacetime $\mathcal{M}$ with boundary $\partial\mathcal{M}$ and denote with $S\mathcal{M}$ the spinor bundle over $\mathcal{M}$. Moreover, let $\Gamma_0(S\Sigma_0) \times \Gamma_0(S\mathcal{M})$ the space of data satisfying the compatibility condition (3.11). Then the map*

$$\Gamma_0(S\Sigma_0) \times \Gamma_0(S\mathcal{M}) \to \Gamma_{sc}(S\mathcal{M})$$

which assign to $(\mathfrak{h}, \mathfrak{f})$ a solution Ψ to the Cauchy problem (2.2) is continuous.

A byproduct of the well-posedness of the Cauchy problem is the existence of Green operators:

Proposition 26. *The Dirac operator is Green hyperbolic, i.e. there exist linear maps, dubbed* advanced/retarded Green operator*, $\mathsf{G}^{\pm} \colon \Gamma_{cc}(S\mathcal{M}) \to \Gamma_{sc}(S\mathcal{M})$ satisfying*

(i) $\mathsf{G}^{\pm} \circ \mathsf{D}\, f = \mathsf{D} \circ \mathsf{G}^{\pm} f = f$ *for all* $f \in \Gamma_{cc}(S\mathcal{M})$;
(ii) $\operatorname{supp}\,(\mathsf{G}^{\pm} f) \subset J^{\pm}(\operatorname{supp}\, f)$ *for all* $f \in \Gamma_{cc}(S\mathcal{M})$,

where $J^{\pm}$ denote the causal future (+) and past (-).

Proof. Let $f \in \Gamma_{cc}(S\mathcal{M})$ and choose $t_0 \in \mathbb{R}$ such that $\operatorname{supp} f \subset J^{+}(\Sigma_{t_0})$. By Theorem 1, there exists a unique solution $\psi(f)$ to the Cauchy problem

$$\begin{cases} \mathsf{D}\psi = f \\ \psi|_{\Sigma_0} = 0 \\ \mathsf{M}\Psi|_{\partial\mathcal{M}} = 0. \end{cases}$$

For $f \in \Gamma_{cc}(S\mathcal{M})$ we set $\mathsf{G}^{+} f := \psi$ and notice that $\mathsf{D} \circ \mathsf{G}^{+} f = \mathsf{D}\psi = f$. Note that by the finite speed of propagation, cf. Proposition 10, $\mathsf{G}^{+} f \in \Gamma_{sc}(S\mathcal{M})$. Moreover, $\mathsf{G}^{+} \circ \mathsf{D}\psi = \mathsf{G}^{+} f = \psi$ which finishes the proof of (i). By Proposition 10, we obtain $\operatorname{supp}\, \mathsf{G}^{+} f \subset J^{+}(f)$ and this conclude the proof of (ii).
The existence of the retarded Green operator G^{-} is proven analogously. □

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