THE CAUCHY PROBLEM OF THE LORENTZIAN DIRAC OPERATOR WITH APS BOUNDARY CONDITIONS

by

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Abstract

We consider the classical Dirac operator on globally hyperbolic manifolds with timelike boundary and show well-posedness of the Cauchy initial-boundary value problem coupled to APS-boundary conditions. This is achieved by deriving suitable energy estimates, which play a fundamental role in establishing uniqueness and existence of weak solutions. Finally, by introducing suitable mollifier operators, we study the differentiability of the solutions. For obtaining smoothness we need additional technical conditions.

Keywords: Cauchy problem, classical Dirac operator, APS boundary condition, globally hyperbolic manifold with timelike boundary.

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1 Introduction

The Dirac operator and spinors were introduced originally by Paul Dirac on Minkowski space in order to find a Lorentz-invariant first order equation with finite speed of propagation that is compatible with the Klein-Gordon operator [37]. Later these concepts were generalized to (pseudo)-Riemannian manifolds and in the 60s Atiyah and Singer [2] showed that the index of the Dirac operator – an analytical quantity of the Dirac operator build out of the dimensions of its kernel and cokernel on 'positive spinors'– is equal to a topological quantity only depending on the underlying compact manifold. Their index theorem generalizes several results e.g. Gauß-Bonnet theorem, Riemann-Roch theorem or Hirzebruch's signature theorem, and it has numerous applications in analysis, geometry, topology, and mathematical physics. Less then 10 years later, this theorem was generalized to compact Riemannian manifolds with boundary in [3], by introducing suitable non-local boundary conditions, dubbed *APS boundary conditions*, which are based on the spectral decomposition of the operator induced on the boundary. While an analog of index theorem for close Lorentzian manifolds is unknown and not to be expected, in [11] Bär and Strohmaier provided an index theorem for the Dirac operator on globally hyperbolic manifolds with spacelike boundary and closed Cauchy surfaces – see also [12] for recent development. This is remarkable since, in this geometrical setting, the Dirac operator is hyperbolic rather than elliptic. However, the operator induced on the boundary is still selfadjoint and elliptic and the Atiyah-Patodi-Singer boundary conditions (which is there a spacelike boundary) still make sense. It was shown in [11] that under these conditions the Dirac operator becomes Fredholm and its index is given formally by the same geometric expression as in the Riemannian case. Besides significant contribution to the index theory, their results were used to provide a rigorous geometric derivation of the gravitational chiral anomaly for a Weyl field [10], but only in spatially compact case.

On the other hand and independent of the above, one could ask for the APS-boundary condition on globally hyperbolic manifolds with timelike boundary where the boundary conditions are prescribed on the (closed) boundary of the Cauchy surfaces. Since APS-boundary conditions are non-local, one might ask whether this can be physically relevant at all. For that, let us point out that the APS index theorem plays a key role in the description of the bulk-edge correspondence of topological insulators, through the cancellation of the time-reversal symmetry anomaly by Witten [55].

As a first step, we aim to investigate the well-posedness of the Cauchy problem for the Dirac operator with APS-boundary conditions on *globally hyperbolic spin manifolds with timelike boundary* (*cf.* Definition 2.1). Concretely this entails that for a certain choice of a time function $(M, g) \simeq (\mathbb{R} \times \Sigma, -N^2 dt^2 + h_t)$, being $N \in C^\infty(\mathbb{R} \times \Sigma)$ strictly positive while $\{(\Sigma, h_t)\}_{t\in\mathbb{R}}$ is a smooth family of Riemannian manifolds with smooth boundary $\partial\Sigma$. Within this setting the Cauchy problem for the Dirac operator D_M requires to consider suitable boundary conditions. In fact, let $D_M = -\gamma_M(e_0)[\nabla^{SM}_{e_0} + iD_t - \frac{n}{2}H]$ be the decomposition (*cf.* equation (2.4)) which relates the Dirac operator D_M over SM to the family $D_\bullet = \{D_t\}_{t\in\mathbb{R}}$ of Dirac operators on $S\Sigma_\bullet = \{S\Sigma_t\}_{t\in\mathbb{R}}$. The presence of a non-empty boundary $\partial\Sigma$ implies that, for each $t \in \mathbb{R}$, D_t is self-adjoint only if appropriate boundary conditions are imposed. Our choice is to require APS boundary conditions, see Definition 2.6 (and for that we have to assume that $\partial\Sigma$ is compact). Per definition this entails that $\psi|_{\partial\Sigma_t}$ belongs to the negative part of spectrum of a self-adjoint first order differential operator A_t built out of D_t (*cf.* Definition 2.5). Loosely speaking, one may think of A_t as the Dirac operator associated with $\partial\Sigma_t$.

We stress that, differently from the wave equation, where the Cauchy problem has been intensively investigated with local and non-local boundary conditions, see e.g. [28, 33, 44, 54], the Cauchy problem for the Dirac operator has been investigated only with local boundary conditions, see e.g. [46, 47]. On the contrary we are here considering boundary conditions which show non-local features (*cf.* Equation (2.6)). As a consequence proving smoothness of the solutions cannot be achieved simply by localizing the problem with suitable partition of unity and usage of appropriate coordinates to reduce the operator to a symmetric hyperbolic system in the classical PDE sense. Therefore, we have to introduce a different strategy to the one employed in [46]. We use the idea in [5, Theorem 3.7.7] for the boundary-less situation using mollifiers to regularize the problem. Differently from the boundaryless case where a suitable mollifier can be constructed for any symmetric hyperbolic system, in this paper we can realize a mollifier only for the Dirac operator with self-adjoint boundary conditions. It would be interesting to extend this analysis to more general Friedrichs systems, including not only symmetric hyperbolic systems, but also symmetric positive systems. As shown in [46], many parabolic PDEs can be rewritten as a positive system, e.g. any diffusion-reaction systems with linear reaction terms. Since our analysis does not require any localization technique, it could be possible to extend the analysis of diffusion-reaction systems with nonlocal mobility which model a wide range of phenomena in biology (see e.g. [40]) and which is up to now only performed on R^n (see e.g. [39]) to manifolds.

Our first main result is the existence and uniqueness of L^2-weak solutions in a time-strip. For the definition of weak solution used here see (3.3).

Theorem 1.1. *Let (M, g) be a globally hyperbolic spin manifold with timelike boundary and let $t : M \to \mathbb{R}$ be a Cauchy temporal function with gradient tangent to the boundary. Assume the boundary of any Cauchy surface is compact. For any $t_a < t_b$ denote with $M_T := t^{-1}[t_a, t_b]$ a time strip with $0 \in [t_a, t_b]$. If $\ker A_t = \{0\}$ for all $t \in \mathbb{R}$, then there exists a unique weak solution to the Cauchy problem*

$$\begin{cases} D_M \psi = f \in \Gamma_{cc}(SM|_{M_T}) \\ \psi|_{\Sigma_0} = \psi_0 \in \Gamma_{cc}(SM|_{\Sigma_0}) \\ \psi \in \operatorname{dom} D_M^{\text{APS}} \end{cases} ,$$

where $\operatorname{dom} D_M^{\text{APS}} := \{\psi \in L^2(SM|_{M_T}) \mid \psi|_{\Sigma_t} \in \operatorname{dom} D_t^{\text{APS}}\}$ *– cf. Equation (2.9). Here $\Sigma_0 = \{0\} \times \Sigma$ while $\Gamma_{cc}(\cdot)$ indicates the space of section compactly supported in the interior of the underlying manifold.*

Let us remark that assuming the boundary of any Cauchy surface to be compact is necessary in order to formulate APS boundary conditions on D_t for all $t \in \mathbb{R}$. The condition $\ker A_t = \{0\}$ can be relaxed by assuming a decomposition $\ker A_t = V_1 \oplus V_2$ which is flipped by the action of $\sigma_{D_t}(e_n)$, that is $\sigma_{D_t}(e_n)V_1 = V_2$.

In order to study the regularity of weak solutions, we need some extra technical assumptions.

Theorem 1.2. *Let $(M, g) = (\mathbb{R} \times \Sigma, -N^2 \mathrm{d}t^2 + h_t)$ be a globally hyperbolic spin manifold with timelike boundary. As for Theorem 1.1, we assume that $\partial\Sigma$ is compact and $\ker A_t = \{0\}$ for all $t \in \mathbb{R}$. If $N|_{\partial M} = 1$ and the unit normal e_n to ∂M is parallel transported along the vector field $e_0 := N^{-1}\partial_t$, then there exists a unique smooth solution $\psi \in \Gamma_{\text{APS}}(SM)$ – cf. Equation (2.6) – to the Cauchy problem*

$$\begin{cases} D_M \psi = f \in \Gamma_{cc}(SM) \\ \psi|_{\Sigma_0} = \psi_0 \in \Gamma_{cc}(SM|_{\Sigma_0}) \\ \psi \in \Gamma_{\text{APS}}(SM) . \end{cases} \tag{1.1}$$

Moreover the problem (1.1) is well-posed in the sense that the linear map

$$\Gamma_{cc}(SM|_\Sigma) \times \Gamma_{cc}(SM) \ni (\psi_0, f) \to \psi \in \Gamma_{\text{APS}}(SM) ,$$

is continuous (see Section 5 for the involved topologies).

Theorem 1.2 in particular requires the rather strong conditions $N|_{\partial M} = 1$ and $\nabla_{e_0} e_n|_{\partial M} = 0$ – later referred to as Assumption 3. However, we want to stress that, to the best of our knowledge, this is the first result showing well-posedness of the Cauchy problem for the Dirac equation with non-local boundary conditions on a class of manifolds which is not conformal to an ultrastatic space-time. These assumptions are used in Section 4 to reduce the problem into a suitable Hamiltonian form. In fact, in the study of the Dirac equation we may then always assume $N = 1$ by applying a conformal transformation: $N|_{\partial M} = 1$ guarantees that such transformation does not spoil the boundary conditions. Similarly, recasting the Cauchy problem (1.1) into an Hamiltonian form requires to identify all Cauchy surfaces $\Sigma_\bullet$ (as well as the associated spinor bundles $SM|_{\Sigma_\bullet}$) by parallel transport along the vector field $e_0 = N^{-1}\partial_t$. Once again, $\nabla_{e_0} e_n|_{\partial M} = 0$ guarantees that the boundary conditions are not messed up in the process.

Showing the well-posedness of the Cauchy problem is not the end of the story: Indeed questions about explicit construction of the evolution operator as in [26, 27] and the quantization of the theory are still to be investigated. Clearly, the well-posedness of the Cauchy problem will guarantee the existence of Green operators (*cf.* Proposition 5.1) which play a pivotal rôle in the algebraic approach to linear quantum field theory, see e.g. [22, 45] for textbooks, [8, 9, 17, 43, 48] for recent reviews, [18–21] for homotopical approaches and [23–25, 30–36] for some applications. However, differently from [46, 47], the Green operator will not have the usual support property due to the non-local behavior of the APS boundary condition. It would be desirable to investigate whether the APS Green operators can be employed in the quantization of fermionic field theories following for instance [28–31, 38, 41, 42, 51].

The paper is organized as follows: Section 2 deals with the geometrical setting of globally hyperbolic spin manifold as well as with the precise definition of APS boundary conditions. In Section 3 we derive a suitable energy inequality for the Dirac operator with APS boundary conditions which is the main ingredient to prove the existence of weak solutions, uniqueness of the solution of the Cauchy problem (1.1) as well as to provide a bound on its support – which reduces to the standard finite speed of propagation in the interior of M. Section 4 is devoted to prove the existence part of Theorem 1.2 by suitably generalizing the proof presented in [5, Theorem 3.7.7]. Finally, using standard arguments, in Proposition 5.1 we prove the existence of Green operators.

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2 Geometrical setting

In this section we collect basic facts and conventions concerning spinors and Dirac operators on Lorentzian manifolds. For a detailed introduction the reader may consult [7, 16].

For a given vector bundle $E \to M$ on a smooth manifold with boundary ∂M we shall denote by $\Gamma(E)$, $\Gamma_{\mathrm{c}}(E)$ *resp.* $\Gamma_{\mathrm{cc}}(E)$ the vector spaces made by smooth sections, compactly supported sections *resp.* sections that have compact support in the interior of M.

2.1 Globally hyperbolic spacetimes with timelike boundary

In the forthcoming discussion (M, g) shall denote an $(n+1)$-dimensional Lorentzian manifold – we shall adopt the $(- + \ldots +)$ signature convention. Within this class, we shall focus on those Lorentzian manifolds which are globally hyperbolic with timelike boundary: These provide a suitable background where to analyze the Cauchy problem for hyperbolic operators.

Definition 2.1. A *globally hyperbolic manifold with timelike boundary* is an $(n+1)$-dimensional, oriented, time-oriented, Lorentzian smooth manifold (M, g) with smooth boundary ∂M such that

(i) the pullback of g with respect to the natural inclusion $\iota\colon \partial M \to M$ defines a Lorentzian metric $\iota^* g$ on the boundary;

(ii) M is causal, i.e. there are no closed causal curves;

(iii) for every point $p, q \in M$, $J^+(p) \cap J^-(q)$ is either empty or compact, where $J^+(p)$ (*resp.* $J^-(p)$) denotes the causal future (*resp.* past) of $p \in M$.

For convenience we recall the following theorem, which extends a known result on globally hyperbolic manifolds without boundaries [13–15].

Theorem 2.2. *[1, Theorem 1.1] Any globally hyperbolic manifold with timelike boundary admits a Cauchy temporal function $t\colon M \to \mathbb{R}$ with gradient tangent to ∂M. This implies that, up to isometric isomorphisms, $M = \mathbb{R} \times \Sigma$ with metric*

$$g = -N^2 \mathrm{d}t^2 + h_t ,$$

where $N \in C^\infty(\mathbb{R} \times \Sigma)$ is a smooth strictly positive function, while $\{h_t\}_{t\in\mathbb{R}}$ is a smooth family of Riemannian metric on Σ. Finally, for all $t \in \mathbb{R}$, $\Sigma_t := \{t\} \times \Sigma$ is a spacelike Cauchy smooth hypersurfaces with boundary $\partial\Sigma_t := \{t\} \times \partial\Sigma$, namely an achronal set intersected exactly once by every inextensible timelike curve.

The scalar products induced by g and h_t will be denoted by $(\ ,\ )_g$, $(\ ,\ )_{h_t}$. For later convenience we set

$$e_0 := N^{-1}\partial_t , \tag{2.1}$$

the globally defined unit normal to the foliation $\Sigma_\bullet := \{\Sigma_t\}_{t\in\mathbb{R}}$ – notice that $(e_0, e_0)_g = -1$. Unless otherwise stated, in what follows a locally defined orthonormal frame $\{e_j\}_{j=0}^n$ will always contain $e_0 = N^{-1}\partial_t$ while e_n shall always refer to the unit normal to $\partial\Sigma_\bullet = \{\partial\Sigma_t\}_{t\in\mathbb{R}}$ in $\Sigma_\bullet$.

2.2 Spin structure and Dirac operator

We shall now assume that (M, g) is a *spin* globally hyperbolic spacetime with timelike boundary. Note that since hyperbolic manifolds are automatically orientable begin spin is not an additional assumption in dimension 4. We will denote by SM the spinor bundle over M – similarly $S\Sigma_t$ denotes the spinor bundle over Σ_t for all $t \in \mathbb{R}$. We denote by $(\ ,\ )_{SM}$ (*resp.* $(\ ,\ )_{S\Sigma_t}$) the indefinite (*resp.* positive definite) pairing over SM (*resp.* over $S\Sigma_t$). The Clifford multiplication $\gamma_M\colon TM \to \mathrm{End}(SM)$ (*resp.* $\gamma_{\Sigma_t}\colon T\Sigma_t \to \mathrm{End}(S\Sigma_t)$) is symmetric (*resp.* skew-symmetric) with respect to $(\ ,\ )_M$ (*resp.* $(\ ,\ )_{\Sigma_t}$), moreover,

$$\gamma_M(X)\gamma_M(Y) + \gamma_M(Y)\gamma_M(X) = -2(X, Y)_g , \tag{2.2}$$

for all $X, Y \in \Gamma(TM)$.

Remark 2.3. We recall that, if n is even, $SM|_{\Sigma_t} = S\Sigma_t$ with $\gamma_M(X) = -i\gamma_M(e_0)\gamma_{\Sigma_t}(X)$ for all $X \in \Gamma(T\Sigma_t)$. Note that this choice differs from the one in [7] by the multiplication by i which is compensated in the definition of the Dirac operator in (2.3) where there is an i in [7]. Moreover in this case $(\ ,\ )_{S\Sigma_t} = (\ , \gamma_M(e_0)\ )_{SM}$. If n is odd then $SM|_{\Sigma_t} = S\Sigma_t^{\oplus 2}$ with $\gamma_M(X) = \begin{pmatrix} 0 & i\gamma_{\Sigma_t}(X) \\ -i\gamma_{\Sigma_t}(X) & 0 \end{pmatrix}$. The corresponding pairings are related by $(\ ,\ )_{S\Sigma_t^{\oplus 2}} = (\ , \gamma_M(e_0)\ )_{SM}$ where $\gamma_M(e_0) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

We shall denote by ∇^{SM}, $\nabla^{S\Sigma_t}$ the spin connection on SM and $S\Sigma_t$ induced by the Levi-Civita connections ∇^M, ∇^{Σ_t} respectively. The *Dirac operator* D_M on SM is defined by

$$D_M := \sum_{j=0}^{n} \varepsilon_j \gamma_M(e_j) \nabla^{SM}_{e_j} , \tag{2.3}$$

where $\{e_j\}_{j=0}^n$ is any locally defined orthonormal frame, while $\varepsilon_j := (e_j, e_j)_g$. Similarly the family $D_\bullet := \{D_t\}_{t\in\mathbb{R}}$ of Dirac operators $D_t = D_{\Sigma_t}$ over $S\Sigma_t$ is defined by $D_t := \sum_{j=1}^n \gamma_{\Sigma_t}(e_j)\nabla^{S\Sigma_t}_{e_j}$. Notice that $D_\bullet$ is a family of elliptic operators.

A standard computation shows the relation between D_M and $D_\bullet$: In particular we have, cp. [7, (3.5)],

$$\nabla^{SM}_X \psi = \nabla^{S\Sigma_\bullet}_X \psi - \frac{1}{2}\gamma_M(e_0)\gamma_M(\nabla^M_X e_0)\psi\,,$$

for all $\psi \in \Gamma(SM)$ and $X \in \Gamma(TM)$ which is tangent to $\Sigma_\bullet$ – that is, $(X, e_0)_g = 0$. A direct inspection leads to

$$D_M = -\gamma_M(e_0)\left[\nabla^{SM}_{e_0} + iD_\bullet - \frac{n}{2}H_\bullet\right]\,, \tag{2.4}$$

where $H_t := -\frac{1}{n}\sum_{j=1}^n (e_j, \nabla^M_{e_j} e_0)_{h_t}$ is the mean curvature of Σ_t. Actually, the latter formula holds for n even, while for n odd we have to replace $D_\bullet$ with $\begin{pmatrix} D_\bullet & 0 \\ 0 & -D_\bullet \end{pmatrix}$.

2.3 APS boundary conditions

In the rest of the paper we shall be interested in solving the Dirac equation $D_M\psi = 0$ – possibly with a suitable source term – where ψ is constrained by appropriate boundary conditions. In fact, we shall require that ψ satisfies the APS boundary condition with respect to the family of operators $D_\bullet$. The following discussion mainly profit of the results of [6, Sec. 2.2]. To fit with the geometrical setting presented therein the following assumption is compulsory.

Assumption 1. We shall assume that $\partial\Sigma$ is compact.

To introduce APS boundary conditions some preparations are in order. We shall denote by $L^2(SM|_{\Sigma_\bullet}) = \{L^2(SM|_{\Sigma_t})\}_{t\in\mathbb{R}}$ the family of L^2-spaces associated with the scalar products

$$(\psi_1, \psi_2)_{L^2(SM|_{\Sigma_t})} := \int_{\Sigma_t} (\psi_1, \gamma_M(e_0)\psi_2)_{SM}\mu_{\Sigma_t}\,,$$

where μ_{Σ_t} is the volume form associated with (Σ_t, h_t). Similarly we shall consider the family of L^2-spaces $L^2(SM|_{\partial\Sigma_\bullet}) = \{L^2(SM|_{\partial\Sigma_t})\}_{t\in\mathbb{R}}$ with scalar product defined by

$$(\varphi_1, \varphi_2)_{L^2(SM|_{\partial\Sigma_t})} = \int_{\partial\Sigma_t} (\varphi_1, \gamma_M(e_0)\varphi_2)_{SM}\mu_{\partial\Sigma_t}\,.$$

Finally we denote by $H^\bullet_{\text{LOC}}(SM|_{\Sigma_\bullet}) = \{H^k_{\text{LOC}}(SM|_{\Sigma_t})\}^{k\in\mathbb{N}}_{t\in\mathbb{R}}$ the family of local Sobolev spaces associated with $L^2(SM|_{\Sigma_\bullet})$.

Remark 2.4. Notice that if n is even, we have $L^2(SM|_{\Sigma_\bullet}) = L^2(S\Sigma_\bullet) = \{L^2(S\Sigma_t)\}_{t\in\mathbb{R}}$, while if n is odd $L^2(SM|_{\Sigma_\bullet}) = L^2(S\Sigma_\bullet^{\oplus 2})$. Here $L^2(S\Sigma_t)$ is the L^2-space with scalar product defined by

$$(\psi_1, \psi_2)_{L^2(S\Sigma_t)} := \int_{\Sigma_t} (\psi_1, \psi_2)_{S\Sigma_t}\mu_{\Sigma_t}\,.$$

A similar comments holds for what concerns $\partial\Sigma_\bullet$. Notice that, depending on the parity of n we may need to consider copies of the latter spaces as

$$S_xM = \begin{cases} S_x\Sigma_\bullet = S_x\partial\Sigma_\bullet^{\oplus 2} & n \in 2\mathbb{Z} \\ S_x\Sigma_\bullet^{\oplus 2} = S_x\partial\Sigma_\bullet^{\oplus 2} & n \notin 2\mathbb{Z} \end{cases} \qquad \forall x \in \partial\Sigma_\bullet\,.$$

This implies that $L^2(SM|_{\partial\Sigma_\bullet}) = L^2(S\partial\Sigma_\bullet^{\oplus 2})$ where the L^2-product on $L^2(S\partial\Sigma_\bullet)$ is given by

$$(\varphi_1, \varphi_2)_{L^2(S\partial\Sigma_t)} = \int_{\partial\Sigma_t} (\varphi_1, \varphi_2)_{S\partial\Sigma_t}\mu_{\partial\Sigma_t}\,.$$

For later convenience we shall now recall the following technical Definition [6, Sec. 2.2]

Definition 2.5. For all $t \in \mathbb{R}$, a formally self-adjoint linear differential operator $A_t \colon \Gamma(SM|_{\partial\Sigma_t}) \to \Gamma(SM|_{\partial\Sigma_t})$ is called *adapted to* D_t if for all $p \in \partial\Sigma_t$ and $\xi \in T^*_p\partial\Sigma_t$

$$\sigma_{A_t}(\xi) = \sigma_{D_t}(e_n^\flat)^{-1} \circ \sigma_{D_t}(\xi)\,, \qquad \sigma_{D_t}(e_n^\flat) \circ A_t = -A_t \circ \sigma_{D_t}(e_n^\flat)\,, \tag{2.5}$$

where $\sigma_{A_t}, \sigma_{D_t}$ denotes the principal symbols of A_t and D_t respectively, while we identified $T^*_p\partial\Sigma_t$ with $\{\xi \in T^*_p\Sigma_t \,|\, \xi(e_n|_p) = 0\}$, e_n being the normalized normal to $\partial\Sigma_t$ and $e_n^\flat := g(e_n, \ ) = h_t(e_n, \ )$.

The existence of A_t is guaranteed by [6, Lemma 2.2] while uniqueness holds up to addition of any $R_t \in \mathrm{End}(SM|_{\partial\Sigma_t})$ which anticommutes with $\sigma_{D_t}(e_n^\flat)$. Notice in particular that, as D_t depends smoothly on t, we shall tune the various choices of A_t in order to get a smooth family $A_\bullet := \{A_t\}_{t\in\mathbb{R}}$ of operators adapted to $D_\bullet$. (Here smoothness merely indicates that all involved operators have coefficients smoothly depending in t.)

Assumption 2. In what follows we shall assume there exists a family $A_\bullet = \{A_t\}_{t\in\mathbb{R}}$ of operators adapted to $D_\bullet = \{D_t\}_{t\in\mathbb{R}}$ such that $\ker A_t = \{0\}$ for all $t \in \mathbb{R}$.

Recalling Assumption 1 one has that, for all $t \in \mathbb{R}$, A_t is a densely defined linear operator on $L^2(SM|_{\partial\Sigma_t})$ which is essentially self-adjoint – *cf.* [6]. Up to considering the closure of such operator – which we still denote by A_t – we get a decomposition of $L^2(SM|_{\partial\Sigma_t})$ into eigenvectors of A_t, namely

$$L^2(SM|_{\partial\Sigma_t}) = \bigoplus_{j\in\mathbb{Z}} \mathbb{C}\varphi_t^{(j)}\,,$$

where $\varphi_t^{(j)}$ is an eigenfunction with corresponding eigenvalues $\lambda_t^{(j)}$.

Let $H^\bullet(A_t) = \{H^s(A_t)\}^{s\in\mathbb{R}}$ be the family of Sobolev spaces associated with A_t – actually $H^s(A_t) := \mathrm{dom}(\langle A_t\rangle^s)$ where $\langle x\rangle := (1+x^2)^{\frac{1}{2}}$. For any subset E of the spectrum of A_t let Π^t_E the corresponding spectral projection and let $H^\bullet_E(A_t) := \{\Pi^t_E H^s(A_t)\}_{t\in\mathbb{R}}$. Finally let $\check{H}(A_t) := H^{\frac{1}{2}}_{(-\infty,0)}(A_t) \oplus H^{-\frac{1}{2}}_{[0,+\infty)}(A_t)$.

Definition 2.6. We denote that space of sections satisfying the *APS boundary conditions* by

$$\Gamma_{\mathrm{APS}}(SM) := \{\psi \in \Gamma(SM) \,|\, \psi|_{\partial\Sigma_\bullet} \in H^{\frac{1}{2}}_{(-\infty,0)}(A_\bullet)\}\,. \tag{2.6}$$

For later convenience – *cf.* the proof of Proposition 3.1 – let us stress that, for any $\psi_1, \psi_2 \in \Gamma_{\mathrm{APS}}(SM)$ we have

$$\int_{\partial\Sigma_t} (\psi_1, \gamma_M(e_n)\psi_2)_{SM}\mu_{\partial\Sigma_t} = 0 \qquad \forall t \in \mathbb{R}\,. \tag{2.7}$$

The following result has been proved in a way more general setting in [6].

Proposition 2.7. *[6, Theorem 3.2-3.11] Let $t \in \mathbb{R}$ be arbitrary but fixed. Let D_t^{MIN} be the closure of the densely defined symmetric operator $D_t|_{\Gamma_{\mathrm{cc}}(SM|_{\Sigma_t})}$ on $L^2(SM|_{\Sigma_t})$ and let $r_t \colon \Gamma_{\mathrm{c}}(SM|_{\Sigma_t}) \to \Gamma_{\mathrm{c}}(SM|_{\partial\Sigma_t})$ be the standard restriction map. Then:*

(i) The adjoint of D_t^{MIN} coincides with the maximal operator D_t^{MAX} defined by

$$\mathrm{dom}\, D_t^{\mathrm{MAX}} := \{\psi \in L^2(SM|_{\Sigma_t}) \,|\, D_t\psi \in L^2(SM|_{\Sigma_t})\}\,, \qquad D_t^{\mathrm{MAX}}\psi := D_t\psi\,. \tag{2.8}$$

(ii) the restriction map r_t extends to a continuous surjection $r_t \colon \mathrm{dom}\, D_t^{\mathrm{MAX}} \to \check{H}(A_t)$, where $\mathrm{dom}\, D_t^{\mathrm{MAX}}$ is equipped with the graph norm of D_t^{MAX} while the norm of $\check{H}(A_t)$ is the one induced by the spaces $H^{\frac{1}{2}}_{(-\infty,0)}(A_t)$, $H^{-\frac{1}{2}}_{[0,+\infty)}(A_t)$. Moreover $\ker r_t = \mathrm{dom}\, D_t^{\mathrm{MIN}}$ so that $\check{H}(A_t) \simeq \frac{\mathrm{dom}\, D_t^{\mathrm{MAX}}}{\mathrm{dom}\, D_t^{\mathrm{MIN}}}$.

(iii) *the operator* D_t^{APS} *defined by*

$$\operatorname{dom} D_t^{\mathrm{APS}} := \{\psi \in \operatorname{dom}(D_t^{\mathrm{MAX}}) \,|\, r_t\psi \in H^{\frac{1}{2}}_{(-\infty,0)}(A_t)\}\,, \qquad D_t^{\mathrm{APS}}\psi := D_t\psi\,, \tag{2.9}$$

is a self-adjoint extension of D_t^{MIN}.

(iv) *denoting with* $\Sigma_t^{\mathrm{INT}} := \Sigma_t \backslash \partial\Sigma_t$ *it holds*

$$\forall\psi \in \operatorname{dom} D_t^{\mathrm{MAX}} \colon \psi \in H^{\bullet+1}_{\mathrm{LOC}}(SM|_{\Sigma_t^{\mathrm{INT}}}) \Longleftrightarrow D_t\psi \in H^{\bullet}_{\mathrm{LOC}}(SM|_{\Sigma_t^{\mathrm{INT}}}) \tag{2.10}$$

$$\forall\psi \in \operatorname{dom} D_t^{\mathrm{APS}} \colon \psi \in H^{\bullet+1}_{\mathrm{LOC}}(SM|_{\Sigma_t}) \Longleftrightarrow D_t\psi \in H^{\bullet}_{\mathrm{LOC}}(SM|_{\Sigma_t})\,, \tag{2.11}$$

where $H^{\bullet}_{\mathrm{LOC}}(SM|_{\Sigma_t^{\mathrm{INT}}}) = \{H^k_{\mathrm{LOC}}(SM|_{\Sigma_t^{\mathrm{INT}}})\}^{k\in\mathbb{N}}$ *is the famliy of local Sobolev spaces associated to* $L^2(SM|_{\Sigma_t^{\mathrm{INT}}})$ *– notice that* $H^{\bullet}_{\mathrm{LOC}}(SM|_{\Sigma_t^{\mathrm{INT}}}) \subset H^{\bullet}_{\mathrm{LOC}}(SM|_{\Sigma_t})$ *is a strict inclusion.*

Remark 2.8. Let us remark that Item *(iii)* requires $\ker A_t = \{0\}$ (Assumption 2). Furthermore notice that for all $\psi_1, \psi_2 \in \Gamma_{\mathrm{APS}}(SM) \cap \Gamma_{\mathrm{c}}(SM)$ it holds using (2.7)

$$\int_M (\psi_1, D_M\psi_2)_{SM}\mu_M + \int_M (D_M\psi_1, \psi_2)_{SM}\mu_M = 0\,,$$

where μ_M is the volume form associated with (M, g).

3 Energy estimates

The aim of this section is to derive suitable energy estimates for the Dirac operator coupled with APS boundary conditions (*cf.* Proposition 3.1). This result will be employed to show that the Cauchy problem (1.1) admits at most one solution (*cf.* Proposition 3.3). In the literature for other boundary conditions, the energy estimates are also employed to show finite speed of propagation. As APS boundary conditions are non-local it is in general not true that a smooth solution ψ of the Cauchy problem (1.1) propagates with at most speed of light, however, Proposition 3.2 will provide a bound on the propagation of the support $\operatorname{supp}(\psi)$ of ψ.

In the following Proposition we deduce an energy estimate for sections $\psi \in \Gamma_{\mathrm{APS}}(SM)$ (*cf.* [4, Theorem 5.3] for the analogous result in the boundaryless case).

Proposition 3.1. *Let* $t_0, t_1 \in \mathbb{R}$, $t_0 < t_1$, *and* $p \in M$ *be such that* $J^-(p) \cap ([t_0, t_1] \times \Sigma) \cap \partial M = \varnothing$. *Then there exists* $C = C_{t_0,t_1} > 0$ *such that for all* $\psi \in \Gamma(SM)$

$$\begin{aligned}\int_{J^-(p)\cap\Sigma_{t_1}} (\psi, \gamma_M(e_0)\psi)_{SM}\mu_{\Sigma_{t_1}} \leqslant Ce^{C(t_1-t_0)} \int_{t_0}^{t_1}\int_{J^-(p)\cap\Sigma_s} (D_M\psi, \gamma_M(e_0)D_M\psi)_{SM}\mu_{\Sigma_s}\mathrm{d}s \\ + e^{C(t_1-t_0)}\int_{J^-(p)\cap\Sigma_{t_0}} (\psi, \gamma_M(e_0)\psi)_{SM}\mu_{\Sigma_{t_0}}\,.\end{aligned} \tag{3.1}$$

Similarly let $K \subset \Sigma$ *be a compact set such that* $\partial\Sigma \subseteq K$ *and let* $K_t := \{t\} \times K$. *Then there is a constant* $C = C_{t_0,t_1} > 0$ *such that for all* $\psi \in \Gamma_{\mathrm{APS}}(SM)$ *it holds*

$$\begin{aligned}\int_{K_{t_1}} (\psi, \gamma_M(e_0)\psi)_{SM}\mu_{\Sigma_{t_1}} \leqslant Ce^{C(t_1-t_0)} \int_{t_0}^{t_1}\int_{J^-(K_{t_1})\cap\Sigma_s} (D_M\psi, \gamma_M(e_0)D_M\psi)_{SM}\mu_{\Sigma_s}\mathrm{d}s \\ + e^{C(t_1-t_0)}\int_{J^-(K_{t_1})\cap\Sigma_{t_0}} (\psi, \gamma_M(e_0)\psi)_{SM}\mu_{\Sigma_{t_0}}\,.\end{aligned} \tag{3.2}$$

Let us remark on the differences between inequalities (3.1) and (3.2). The main difference lies on the fact that in (3.1) we integrate over domains which have empty intersection with ∂M. On the contrary, the domains used in (3.2) are such that they contain $(t_1, t_0) \times \partial\Sigma$ entirely. Intermediate situations, where $(t_1, t_0) \times \partial\Sigma$ only partially intersects with the domain of integration domain are not feasible: This is due to the non-locality of the APS boundary conditions – *cf.* Definition 2.6.

Proof of Proposition 3.1. The proof of inequalities (3.1) and (3.2) is similar and is based on standard techniques (*cf.* [5]). We shall focus on inequality (3.2) as this is the one where the boundary conditions enters. For that, let $\psi \in \Gamma_{\mathrm{APS}}(SM)$ and let $V_\psi \in \Gamma(TM)$ be the vector field defined by

$$(V_\psi, X)_g := (\psi, \gamma_M(X)\psi)_{SM}\,, \qquad \forall X \in \Gamma(TM)\,.$$

By direct inspection one has

$$\mathrm{div}_M(V_\psi) = \sum_{j=0}^{n} \varepsilon_j (e_j, \nabla^M_{e_j} V_\psi) = (D_M\psi, \psi)_{SM} + (\psi, D_M\psi)_{SM}\,.$$

Let us now consider $U := J^-(K_{t_1}) \cap ((t_0, t_1) \times \Sigma)$ and integrate the previous equality on U with respect to μ_M. We get

$$\begin{aligned}
\int_U \mathrm{div}_M(V_\psi)\mu_M &= \int_U [(D_M\psi, \psi)_{SM} + (\psi, D_M\psi)_{SM}]\mu_M \\
&\leqslant \int_U [C_1(D_M\psi, \gamma_M(e_0)D_M\psi)_{SM} + C_2(\psi, \gamma_M(e_0)\psi)_{SM}]\mu_M \\
&= \int_{t_0}^{t_1}\int_{J^-(K_{t_1})\cap\Sigma_s} [C_1(D_M\psi, \gamma_M(e_0)D_M\psi)_{SM} + C_2(\psi, \gamma_M(e_0)\psi)_{SM}]\mu_{\Sigma_s}\mathrm{d}s\,,
\end{aligned}$$

where $C_1, C_2 > 0$ do not depend on ψ. Here, in the second line the inequality uses $(D_M\psi, \psi)_{SM} = (D_M\psi, \gamma_M(e_0)\gamma_M(e_0)\psi)_{SM}$, Cauchy-Schwarz for the positive definite pairing $(, \gamma_M(e_0))_{SM}$ and that $\gamma_M(e_0)$ is a bounded operator with respect to the fiberwise scalar product $(\psi_1, \gamma_M(e_0)\psi_2)_{SM}$. By Stokes' theorem, the left-hand side of the equation equals to

$$\int_U \mathrm{div}_M(V_\psi)\mu_M = \int_{\partial U} \iota^*_{\partial U}(\iota_{V_\psi}\mu_M)\,.$$

In fact, since $\partial\Sigma \subset \partial K$ we have $\partial U = K_{t_1} \cup (J^-(K_{t_1}) \cap \Sigma_{t_0}) \cup ((t_0, t_1) \times \partial\Sigma) \cup (\partial J^-(K_{t_1}) \cap ((t_0, t_1) \times (\Sigma\backslash\partial\Sigma)))$ so that

$$\begin{aligned}
\int_U \mathrm{div}_M(V_\psi)\mu_M = &\int_{K_{t_1}} (\psi, \gamma_M(e_0)\psi)_{SM}\mu_{\Sigma_{t_1}} - \int_{J^-(K_{t_1})\cap\Sigma_{t_0}} (\psi, \gamma_M(e_0)\psi)_{SM}\mu_{\Sigma_{t_0}} \\
&- \int_{t_0}^{t_1}\int_{\partial\Sigma_s} (\psi, \gamma_M(e_n)\psi)_{SM}\mu_{\Sigma_s}\mathrm{d}s + \int_{\partial J^-(K_{t_1})\cap((t_0,t_1)\times(\Sigma\backslash\partial\Sigma))} \iota^*_{\partial U}(\iota_{V_\psi}\mu_M)\,.
\end{aligned}$$

The integral over $\partial\Sigma$ vanishes on account of the boundary condition of $\psi \in \Gamma_{\mathrm{APS}}(SM)$ – *cf.* Equation (2.7). Moreover, since $(\ , \gamma_M(X)\ )_{SM}$ is positive definite for all future-pointed time-like vectors X, the same holds true for future-pointed light-like vectors. This entails that

$$\int_{\partial J^-(K_{t_1})\cap((t_0,t_1)\times(\Sigma\backslash\partial\Sigma))} \iota^*_{\partial U}(\iota_{V_\psi}\mu_M) = \int_{\partial J^-(K_{t_1})\cap((t_0,t_1)\times(\Sigma\backslash\partial\Sigma))} (\psi, \gamma_M(X)\psi)\iota_X\mu_M \geqslant 0\,,$$

where X is some future-directed, light-like and tangent to $\partial J^-(K_{t_1})$. Overall we have

$$\begin{aligned}
&\int_{K_{t_1}} (\psi, \gamma_M(e_0)\psi)_{SM}\mu_{\Sigma_{t_1}} - \int_{J^-(K_{t_1})\cap\Sigma_{t_0}} (\psi, \gamma_M(e_0)\psi)_{SM}\mu_{\Sigma_{t_0}} \\
&\qquad\leqslant C\int_{t_0}^{t_1}\int_{J^-(K_{t_1})\cap\Sigma_s} [(D_M\psi, \gamma_M(e_0)D_M\psi)_{SM} + (\psi, \gamma_M(e_0)\psi)_{SM}]\mu_{\Sigma_s}\mathrm{d}s\,,
\end{aligned}$$

which leads inequality (3.2) by Gronwall's Lemma applied to the function

$$F(t) := \int_{J^-(K_{t_1})\cap\Sigma_t} (\psi, \gamma_M(e_0)\psi)_{SM}\mu_{\Sigma_t}\,.$$

This concludes our proof. □

We are now in the position to prove Theorem 1.1—the existence of weak solution: Let us recall that a weak solution ψ is an element $\psi \in L^2(SM_T)$ such that

$$(\phi, f)_{L^2(SM|_{M_T})} = (D_M^\dagger \phi, \psi)_{L^2(SM|_{M_T})} \tag{3.3}$$

for all $\phi \in \Gamma_c(SM) \cap \operatorname{dom} D_M^{\mathrm{APS}}$, where $L^2(SM|_{M_T})$ is as defined on page 6, μ_g being the canonical volume form of (M_T, g), $e_0 := g(\partial_t, \partial_t)^{-\frac{1}{2}}\partial_t$ and $D_M^\dagger = -D_M$ denotes the formal adjoint of $D_M|_{\Gamma_{\mathrm{cc}}(SM|_{M_T})}$. (This definition uses that the APS boundary condition are self-adjoint. Otherwise ϕ would need to obey the corresponding adjoint boundary condition.)

Proof of Theorem 1.1. Let M_T be a time strip as in Theorem 1.1. Using Proposition 3.1 for $\psi \in \Gamma_c(SM|_{M_T}) \cap \Gamma_{\mathrm{APS}}(SM|_{M_T})$ such that $\psi_{t_a} = 0$ and $\psi_{t_b} = 0$, there exists a constant C such that

$$\|\psi\|_{L^2(SM|_{M_T})}\| \leqslant C\|D_M^\dagger \psi\|_{L^2(SM|_{M_T})} \,.$$

Finally, using the same argument in [47, Theorem 3.20] or [46, Theorem 4.2] we obtain the existence of weak solutions. □

We conclude this section with some property of the support of smooth solutions of the Cauchy problem.

Proposition 3.2 (Support property)**.** *Assume the setup of Theorem 1.2. Any solution $\psi \in \Gamma_{\mathrm{APS}}(SM)$ of the Cauchy problem (1.1) satisfies*

$$\operatorname{supp}\psi \subset J(\operatorname{supp} f \cup \operatorname{supp}\psi_0) \cup J(\partial\Sigma_0)\,, \tag{3.4}$$

where $J(A) := J^+(A) \cup J^-(A)$ denotes the union of the causal future and the causal past of a set A.

Proof. Let $\mathcal{V} := J(\operatorname{supp} f \cup \operatorname{supp}\psi_0) \cup J(\partial\Sigma_0)$ and let $p \in M\backslash\mathcal{V}$ and let assume $p \in J^+(\Sigma_0)$ – the case $p \in J^-(\Sigma_0)$ can be treated analogously. This means that there is no future-directed causal curve contained in $J^+(\Sigma_0)$ which starts in $\mathcal{V}$ and terminates at p. Hence $J^-(p) \cap J^+(\Sigma_0) \cap \operatorname{supp}\mathcal{V} = \varnothing$. By inequality (3.1), $\psi = 0$ on $J^-(p) \cap J^+(\Sigma_0)$, in particular it vanishes as p. □

As byproduct of the energy estimate (3.1) we get uniqueness of smooth solutions.

Proposition 3.3 (Uniqueness)**.** *Assume the setup of Theorem (1.2). Then there exists at most a unique solution $\psi \in \Gamma_{\mathrm{APS}}(SM)$ to the Cauchy problem (1.1).*

Proof. By linearity of the Cauchy problem (1.1), it suffices to prove that $\psi = 0$ is the only solution to the Cauchy problem (1.1) with vanishing initial data. Proposition 3.2 entails that $\operatorname{supp}(\psi) \subseteq J(\Sigma_0)$. Let $p \in \Sigma_t \subset J^+(\Sigma_0)$ – a similar argument applies if p lies in $J^-(\Sigma_0)$ – and let $K_p \subset \Sigma_t$ be a compact set containing $\partial\Sigma_t \cup \{p\}$. Then inequality (3.2) shows that $\int_{K_p} (\psi, \gamma_M(e_0)\psi)_{SM}\mu_{\Sigma_t} = 0$. The arbitrariness of p implies that $\psi = 0$. □

4 Existence of smooth solutions

In this section we shall prove Theorem 1.2 and to this end, we assume the following:

Assumption 3. We shall assume that $N|_{\partial M} = 1$ and $\nabla_{e_0} e_n|_{\partial M} = 0$, that is, e_n is parallel transported along the integral curves of e_0.

Our strategy is divided into two main steps. In the first one we shall prove Theorem 1.2 under the additional assumption that Σ is compact. This will be achieved by reducing the Cauchy problem (1.1) into an Hamiltonian form (*cf.* Equation (4.6): For that Assumption 3 will be crucial). Once in Hamiltonian form, the existence of a smooth solution to the Cauchy problem will be proved by mimicking the proof presented in [5] for the case of symmetric hyperbolic systems on globally hyperbolic manifolds without boundary. Such proof can be suitably modified in the case of a boundary by exploiting the functional calculus associated with the self-adjoint operator D_t^{APS}.

In the second step we shall drop the compactness assumption on Σ. This can be done again along the lines of [5], where the proof is based on finite speed propagation of the solution. In fact, in our case the result of Proposition 3.2 will suffice for our purposes, together with the results obtained in [47].

4.1 Reduction to the Hamiltonian form

In this section we shall reduce the Dirac equation to its Hamiltonian form. To this end, we shall first perform a conformal transformation and then identify a family of spinor bundles. Finally, we shall compare the Cauchy problems for the Dirac operator with the ones for the Hamiltonian form.

4.1.1 Conformal transformation

We start by reducing the Cauchy problem (1.1) to the case where $N = 1$. This is achieved by a conformal transformation $\widehat{g} = N^{-2}g$. The corresponding Clifford multiplication $\widehat{\gamma}_M$ and spin-connection $\widehat{\nabla}^{SM}$ are related to the previous one by

$$\widehat{\gamma}_M(X) = N^{-1}\gamma_M(X)\,, \qquad \widehat{\nabla}^{SM}_X = \nabla^{SM}_X + \frac{N}{2}\gamma_M(X)\gamma_M(\nabla^M N^{-1}) - \frac{N}{2}X(N^{-1})\,,$$

while the fiber pairing $(\ ,\ )_{SM}$ remains unaffected. The Dirac operator $\widehat{D}_M$ for $(M,\widehat{g})$ is related to the Dirac operator D_M for (M,g) by

$$\widehat{D}_M = N^{\frac{n+1}{2}} D_M N^{-\frac{n-1}{2}}\,. \tag{4.1}$$

Notice that $\widehat{e}_j = Ne_j$ and therefore Equations (2.4) and (4.1) entail that $\widehat{D}_\bullet = N_\bullet^{\frac{n+1}{2}} D_\bullet N_\bullet^{-\frac{n-1}{2}}$. This shows that $D_M\psi = 0$ is equivalent to $\widehat{D}_M\widehat{\psi} = 0$, where $\widehat{\psi} = N_\bullet^{\frac{n-1}{2}}\psi$. Moreover, the APS boundary conditions for $D_\bullet$ coincides with the APS boundary conditions for $\widehat{D}_\bullet$ since $N = 1$ at the boundary.

Overall, under Assumption 3 we may reduce the Cauchy problem (1.1) to the case $N = 1$. In what follows we shall implicitly assume that such a reduction has been made. As a consequence the vector field e_0 is geodesic, that is $\nabla_{e_0}e_0 = 0$, as

$$\begin{aligned} 2(\nabla_{e_0}e_0, e_0)_g &= \nabla_{e_0}(e_0, e_0)_g = 0\,, \\ (\nabla_{e_0}e_0, e_j)_g &= -(e_0, \nabla_{e_0}e_j)_g = -(e_0, \nabla_{e_j}e_0)_g + (e_0, [e_0, e_j])_g = 0\,, \qquad j \geqslant 1\,, \end{aligned}$$

where in the last equality we used the hypothesis $N|_{\partial M} = 1$ to ensure that we can choose the local coordinate fields e_j so that $[e_0, e_j] = 0$.

Remark 4.1. One may wonder what happens if the assumption $N|_{\partial M} = 1$ is dropped. In this case we may choose A_t and $\widehat{A}_t$ so that $\widehat{A}_t = N_t^{\frac{n+1}{2}} A_t N_t^{-\frac{n-1}{2}}$, therefore, the connection between $\widehat{A}_t$-APS boundary conditions and A_t-APS boundary conditions is lost. It would be interesting to see whether the A_t-APS boundary induces $\widehat{A}_t$-boundary conditions for which we can still apply proposition 2.7 – *cf.* [6, Theorem 3.2-3.11].

4.1.2 Identification of $\Sigma_\bullet$

After reduction to the case $N = 1$ we shall proceed by identifying all Cauchy surface $\Sigma_\bullet = \{\Sigma_t\}_{t\in\mathbb{R}}$ (and the associated spinor bundles) by e_0-parallel transport. This allows to formulate the Cauchy problem (1.1) in Hamiltonian form, where the unknown spinor $\widetilde{\psi}$ belongs to the space $C^\infty(\mathbb{R}, \Gamma(SM|_\Sigma))$. Within this setting we shall then be able to mimic the proof of [5].

The following discussion profits of the results presented in [53]. We start by identifying the family $SM|_{\Sigma_\bullet}$ with $SM|_\Sigma$ by parallel transport. This can be done as follows: for an arbitrary but fixed $t \in \mathbb{R}$, let $\tau_t^0 \colon \Sigma_t \to \Sigma$ denote the translation $(t, x) \to (0, x)$. Moreover, let $\wp_t^0 \colon SM|_{\Sigma_t} \to SM|_\Sigma$ be the parallel transport which lifts τ_t^0, that is, the parallel transport along the integral curves of e_0. Notice that $\wp_t^0$ preserves $(\, ,\,)_{SM}$ namely $(\psi_1, \psi_2)_{SM} = [\tau_t^0]^*(\wp_t^0\psi_1, \wp_t^0\psi_2)_{SM}$. Furthermore, it also preserves the positive definite form $(\, , \gamma_M(e_0)\,)_{SM}$ because e_0 is geodesic.

Let now $\rho \colon M \to \mathbb{R}$ be such that, for all $t \in \mathbb{R}$, $[\tau_0^t]^*\mu_{\Sigma_t} = \rho_t^2 \mu_\Sigma$ where $\tau_0^t \tau_t^0 = 1$. Then, $\rho \in C^\infty(M)$. We set

$$U \colon \Gamma(SM) \to C^\infty(\mathbb{R}, \Gamma(SM|_\Sigma)), \qquad (U\psi)_t := \rho_t \wp_t^0 \psi \,. \tag{4.2}$$

Notice that U is an isometry between $L^2(SM|_{\Sigma_\bullet})$ and $L^2(SM|_\Sigma)$ as

$$\begin{aligned} ((U\psi_1)_t, (U\psi_2)_t)_{L^2(SM|_\Sigma)} &= \int_\Sigma (\wp_t^0\psi_1, \gamma_M(e_0)\wp_t^0\psi_2)_{SM} \rho_t^2 \mu_\Sigma \\ &= \int_{\Sigma_t} (\psi_1, \gamma_M(e_0)\psi_2)_{SM} \mu_{\Sigma_t} = (\psi_1, \psi_2)_{L^2(SM|_{\Sigma_t})} \,. \end{aligned}$$

We shall now consider the differential operator UD_MU^{-1}. Direct calculation leads to

$$\partial_t (U\psi)_t = (U \rho_t^{-1} \nabla^{SM}_{\partial_t}(\rho_t \psi))_t \,,$$

which entails the identity

$$U \nabla^{SM}_{e_0} U^{-1} = \rho_t \partial_t \circ \rho_t^{-1} = \partial_t - \rho_t^{-1}[\partial_t, \rho_t] \,,$$

where $|h_t| := \det h_t$. At the same time, $\nabla_{e_0} e_0 = 0$ entails that

$$-nH_t = \operatorname{div}_M(e_0) = |h_t|^{-\frac{1}{2}} \partial_t |h_t|^{\frac{1}{2}} = 2\rho_t^{-1}(\partial_t \rho_t) = 2\rho_t^{-1}[\partial_t, \rho_t] \,.$$

Overall we have, using (2.4),

$$UD_MU^{-1} = -\gamma_M(e_0) U \left[\nabla^{SM}_{e_0} + iD_\bullet - \frac{n}{2} H_\bullet \right] U^{-1} = -\gamma_M(e_0) \left[\partial_t + iUD_\bullet U^{-1}\right] \,.$$

It follows that $D_M \psi = f$ is equivalent to

$$\left(\partial_t + i\widetilde{D}_t\right) \widetilde{\psi} = \widetilde{f} \,, \tag{4.3}$$

being $\widetilde{\psi} := U\psi \in C^\infty(\mathbb{R}, \Gamma(SM|_\Sigma))$, $\widetilde{f} := \gamma_M(e_0)Uf \in C^\infty_{\mathrm{c}}(\mathbb{R}, \Gamma_{\mathrm{cc}}(SM|_\Sigma))$ while $\widetilde{D}_\bullet := UD_\bullet U^{-1}$. Notice that, for all $t \in \mathbb{R}$, $\widetilde{D}_t|_{\Gamma_{\mathrm{cc}}(SM|_\Sigma)}$ is a densely-defined operator on $L^2(SM|_\Sigma)$ with symmetric closure. Therefore, as expected, the Cauchy problem associated with Equation (4.3) requires appropriate boundary conditions. In fact, the results [6, Theorem 3.2-3.11] presented in the particular case of Proposition 2.7 also hold true for the family $\widetilde{D}_t$, in particular:

1. $\widetilde{D}_t|_{\Gamma_{\mathrm{cc}}(SM|_{\partial\Sigma})}$ is a densely defined linear operator on $L^2(SM|_\Sigma)$ with symmetric closure $\widetilde{D}_t^{\mathrm{MIN}}$ and adjoint $\widetilde{D}_t^{\mathrm{MAX}}$ defined on the domain

$$\operatorname{dom} \widetilde{D}_t^{\mathrm{MAX}} := \{\psi \in L^2(SM|_\Sigma) \,|\, \widetilde{D}_t\psi \in L^2(SM|_\Sigma)\} \,, \qquad \widetilde{D}_t^{\mathrm{MAX}}\psi = \widetilde{D}_t\psi \,.$$

2. the restriction map $r \colon \Gamma(SM|_\Sigma) \to \Gamma(SM|_{\partial\Sigma})$ extends to a continuous surjection

$$\widetilde{r}_t \colon \operatorname{dom} \widetilde{D}_t^{\mathrm{MAX}} \to \check{H}(\widetilde{A}_t) \,.$$

Here $\widetilde{A}_t$ is a differential operator on $\Gamma(SM|_{\partial\Sigma})$ adapted to $\widetilde{D}_t$, while $\check{H}(\widetilde{A}_t)$ is defined as $\check{H}(A_t)$.

3. the operator $\widetilde{D}_t^{\text{APS}}$ defined by

$$\operatorname{dom}\widetilde{D}_t^{\text{APS}} := \{\psi \in \operatorname{dom}\widetilde{D}_t^{\text{MAX}} \mid \widetilde{r}_t\psi \in H^{\frac{1}{2}}_{(-\infty,0)}(\widetilde{A}_t)\}\,, \qquad \widetilde{D}_t^{\text{APS}}\psi := \widetilde{D}_t\psi\,,$$

is a self-adjoint extension of $\widetilde{D}_t^{\text{MIN}}$ and moreover

$$\forall\psi \in \operatorname{dom}\widetilde{D}_t^{\text{MAX}}\colon \psi \in H^{\bullet+1}_{\text{LOC}}(SM|_{\Sigma^{\text{INT}}}) \Longleftrightarrow \widetilde{D}_t\psi \in H^{\bullet}_{\text{LOC}}(SM|_{\Sigma^{\text{INT}}}) \tag{4.4}$$

$$\forall\psi \in \operatorname{dom}\widetilde{D}_t^{\text{APS}}\colon \psi \in H^{\bullet+1}_{\text{LOC}}(SM|_{\Sigma}) \Longleftrightarrow \widetilde{D}_t\psi \in H^{\bullet}_{\text{LOC}}(SM|_{\Sigma})\,, \tag{4.5}$$

These results call for a comparison between the Cauchy problem (1.1) and the Cauchy problem

$$(\partial_t + i\widetilde{D}_\bullet)\widetilde{\psi} = \widetilde{f}\,, \qquad \widetilde{\psi}_0 = \psi_0\,, \qquad \widetilde{\psi} \in C^\infty(\mathbb{R}, \Gamma_{\text{APS}}(SM|_\Sigma))\,, \tag{4.6}$$

where $\widetilde{\psi} = U\psi$, $\widetilde{f} = \gamma_M(e_0)Uf$ while we denoted

$$C^\infty(\mathbb{R}, \Gamma_{\text{APS}}(SM|_\Sigma)) := \{\psi \in C^\infty(\mathbb{R}, \Gamma(SM|_\Sigma)) \mid \widetilde{r}_t\psi_t \in H^{\frac{1}{2}}_{(-\infty,0)}(\widetilde{A}_t)\,,\ \forall t \in \mathbb{R}\}\,.$$

Finally, for what concern boundary conditions for the Cauchy problems (1.1) and (4.6), we observe that, on account of Assumption 3, we may set $\widetilde{A}_t = UA_tU^{-1}$. Notice that this preserves Assumption 2, namely $\ker\widetilde{A}_t = \{0\}$ for all $t \in \mathbb{R}$. A direct inspection shows that if $\psi \in \Gamma_{\text{APS}}(SM)$ then $U\psi \in C^\infty(\mathbb{R}, \Gamma_{\text{APS}}(SM|_\Sigma))$.

4.2 Compact Cauchy surfaces and smooth solutions

In this section we shall prove that the Cauchy problem (4.6) has a (unique) solution $\widetilde{\psi} \in C^\infty(\mathbb{R}, \Gamma_{\text{APS}}(SM|_\Sigma))$ under the following additional assumption.

Assumption 4. We shall assume that Σ is compact.

4.2.1 Sobolev spaces

For later convenience we shall introduce the family of APS-Sobolev spaces $H^\bullet_{\text{APS},\bullet}(SM|_\Sigma) := \{H^k_{\text{APS},t}(SM|_\Sigma)\}^{k\in\mathbb{N}}_{t\in\mathbb{R}}$, where $H^k_{\text{APS},t}(SM|_\Sigma) := \operatorname{dom}(\langle\widetilde{D}_t^{\text{APS}}\rangle^k)$ – here $\langle z\rangle := (1+z^2)^{\frac{1}{2}}$. As usual $H^\infty_{\text{APS},t}(SM|_\Sigma) := \cap_{k\in\mathbb{N}} H^k_{\text{APS},t}(SM|_\Sigma)$. We recall that $H^k_{\text{APS},t}(SM|_\Sigma)$ can be characterized in terms of the spectral calculus associated with $\widetilde{D}_t^{\text{APS}}$. Indeed, if $P^{(\widetilde{D}_t^{\text{APS}})}$ denotes the projection-valued measure associated with $\widetilde{D}_t^{\text{APS}}$ then [52, Theorem 9.13]

$$H^k_{\text{APS},t}(SM|_\Sigma) = \{\psi \in L^2(SM|_\Sigma) \mid \int_{\mathbb{R}} \langle\lambda\rangle^{2k} \mathrm{d}\mu_\psi(\lambda) < +\infty\}\,, \tag{4.7}$$

where $\mu_\psi(E) = (\psi, P_E^{(\widetilde{D}_t^{\text{APS}})}\psi)_{L^2(SM|_\Sigma)}$ for any Borel set E of $\mathbb{R}$.

Notice that $H^k_{\text{APS},t}(SM|_\Sigma) \hookrightarrow H^k(SM|_\Sigma)$ is a closed inclusion for all $k \in \mathbb{N}$ and $t \in \mathbb{R}$ (*cf.* Proposition 2.7). (As Σ is compact, we shall now drop the subscript $_{\text{LOC}}$ to the Sobolev space $H^k(SM|_\Sigma)$.) In fact, under Assumption 4 (Σ being compact) and being $\widetilde{D}_t$ an elliptic first order differential operator, the norm $\|\ \|_{H^k(SM|_\Sigma)}$ can be taken to coincide with $\|\ \|_{H^k_{\text{APS},t}(SM|_\Sigma)}$ for all $t \in \mathbb{R}$ – that is $\|\psi\|^2_{H^k(SM|_\Sigma)} = (\psi, (1+\widetilde{D}_t^2)^k\psi)_{L^2(SM|_\Sigma)}$. This also shows that any pseudo-differential operator D_ℓ of order ℓ induces a continuous map $D_\ell\colon H^{k+\ell}_{\text{APS},t}(SM|_\Sigma) \to H^k(SM|_\Sigma)$ as

$$\|D_\ell\psi\|_{H^k(SM|_\Sigma)} \leqslant C_{k,\ell}\|\psi\|_{H^{k+\ell}(SM|_\Sigma)} = C_{k,\ell}\|\psi\|_{H^{k+\ell}_{\text{APS},t}(SM|_\Sigma)}\,.$$

Moreover, as $H^{k+\ell}(SM|_\Sigma) \hookrightarrow H^k(SM|_\Sigma)$ is compact for all $\ell, k \in \mathbb{N}$, then $H^{k+\ell}_{\text{APS},t}(SM|_\Sigma) \hookrightarrow H^k_{\text{APS},t}(SM|_\Sigma)$ is compact as well – here we are again using Assumption 4.

For $\ell, k \in \mathbb{N} \cup \{\infty\}$, we denote by $C^\ell(\mathbb{R}, H^k(SM|_\Sigma))$ the space of functions $u: \mathbb{R} \to H^k(SM|_\Sigma)$ which are C^ℓ with respect to one (hence any) of the equivalent norm on $H^k(SM|_\Sigma)$.

Similarly we shall denote by $C^\ell(\mathbb{R}, H^k_{\text{APS},\bullet}(SM|_\Sigma))$ the subspace of $C^\ell(\mathbb{R}, H^k(SM|_\Sigma))$ made of functions u such that $u_t \in H^k_{\text{APS},t}(SM|_\Sigma)$ for all $t \in \mathbb{R}$. If $t \to u_t$ is compactly supported we shall write $u \in C^\ell_{\text{c}}(\mathbb{R}, H^k_{\text{APS},\bullet}(SM|_\Sigma))$. Notice that, for all interval $I \subseteq \mathbb{R}$ and $k, \ell \in \mathbb{N}$, $C^\ell(I, H^k_{\text{APS},\bullet}(SM|_\Sigma))$, is a Fréchet space with seminorms defined by

$$\|u\|_{\tau,\ell} := \sup_{\substack{t \in I \\ 0 \leqslant j \leqslant \ell}} \|\partial_t^j u_t\|_{H^k_{\text{APS},t}(SM|_\Sigma)} .$$

4.2.2 Mollifiers

For $\varepsilon > 0$ we shall denote by $J_t^{(\varepsilon)}$ the linear bounded self-adjoint operator on $L^2(SM|_\Sigma)$ defined by $J_t^{(\varepsilon)} := e^{-\varepsilon\langle \widetilde{D}_t^{\text{APS}}\rangle}$. Notice that $J_t^{(\varepsilon)}\psi \in H^\infty_{\text{APS},t}(SM|_\Sigma)$, in fact, for all $k, \ell \in \mathbb{N}$, $J_t^{(\varepsilon)}: H^k_{\text{APS},t}(SM|_\Sigma) \to H^\ell_{\text{APS},t}(SM|_\Sigma)$ is bounded. Indeed, this follows from (4.7) together with estimate

$$\|J_t^{(\varepsilon)}\psi\|^2_{H^\ell_{\text{APS},t}(SM|_\Sigma)} = \int_{\mathbb{R}} \langle\lambda\rangle^{2\ell} e^{-2\varepsilon\langle\lambda\rangle} \mathrm{d}\mu_\psi(\lambda) \leqslant c_{k,\ell}(\varepsilon) \|\psi\|^2_{H^k_{\text{APS},t}(SM|_\Sigma)} , \tag{4.8}$$

being $c_{k,\ell}(\varepsilon) := \sup_{\lambda\in\mathbb{R}} \langle\lambda\rangle^{2(\ell-k)} e^{-2\varepsilon\langle\lambda\rangle} \propto \varepsilon^{2(k-\ell)}$. In fact, a dominated convergence argument shows that, for all $k \in \mathbb{N}$, $J_t^{(\varepsilon)}: H^k_{\text{APS},t}(SM|_\Sigma) \to H^k_{\text{APS},t}(SM|_\Sigma)$ converges strongly to the identity as $\varepsilon \to 0^+$. This is equivalent to $\lim_{\varepsilon\to 0^+} \|J_t^{(\varepsilon)}\psi - \psi\|_{H^k_{\text{APS},t}(SM|_\Sigma)} = 0$ for all $\psi \in H^k_{\text{APS},t}(SM|_\Sigma)$. (In fact, $J_t^{(\varepsilon)}: H^{k+\ell}_{\text{APS},t}(SM|_\Sigma) \to H^k_{\text{APS},t}(SM|_\Sigma)$ converges strongly to the embedding $H^{k+\ell}_{\text{APS},t}(SM|_\Sigma) \hookrightarrow H^k_{\text{APS},t}(SM|_\Sigma)$ as $\varepsilon \to 0^+$.) Moreover, since $e^{-2\varepsilon\langle\lambda\rangle} \leqslant 1$, Equation (4.8) shows that

$$\|J_t^{(\varepsilon)}\|_{B(H^k_{\text{APS},t}(SM|_\Sigma))} \leqslant 1 ,$$

where $\| \ \|_{B(H^k_{\text{APS},t}(SM|_\Sigma))}$ denotes the operator norm in the space $B(H^k_{\text{APS},t}(SM|_\Sigma))$ of bounded operators on $H^k_{\text{APS},t}(SM|_\Sigma)$.

4.2.3 Existence of smooth solution with compact Cauchy surfaces

The existence of smooth solutions is performed in 3 steps. We first regularize the problem by using the mollifiers from the last subsection and show existence of solutions of these regularized problems. In the second step we show that the regularized solutions have a convergent subsequence. Then, in the third step we prove that the corresponding limit is actually a smooth solution of our Cauchy problem (for compact Cauchy surfaces).

1. *Regularized problem:* In order to prove the existence of a solution $\widetilde{\psi} \in C^\infty(\mathbb{R}, \Gamma_{\text{APS}}(SM|_\Sigma))$ of the Cauchy problem (4.6), we shall first consider the following regularized problem:

$$(\partial_t + iJ_t^{(\varepsilon)} \widetilde{D}_t^{\text{APS}} J_t^{(\varepsilon)}) \widetilde{\psi}_t^{(\varepsilon)} = \widetilde{f}_t , \qquad \widetilde{\psi}_0^{(\varepsilon)} = \psi_0 . \tag{4.9}$$

   Notice that, by assumption $\psi_0 \in \Gamma_{\text{cc}}(SM|_\Sigma) \subset H^\infty_{\text{APS},0}(SM|_\Sigma)$ and $\widetilde{f} \in C^\infty_{\text{c}}(\mathbb{R}, H_{\text{APS},\bullet}(SM|_\Sigma))$ because $f \in \Gamma_{\text{cc}}(SM)$. We shall now prove existence and uniqueness of the solution $\widetilde{\psi}^{(\varepsilon)}$ of (4.9) and we will see that $\widetilde{\psi}^{(\varepsilon)} \in C^1(\mathbb{R}, H^\infty_{\text{APS},\bullet}(SM|_\Sigma))$. For that, let $k \in \mathbb{N}$ be arbitrary but fixed and let observe that

$$t \mapsto \|J_t^{(\varepsilon)} \widetilde{D}_t^{\text{APS}} J_t^{(\varepsilon)}\|_{B(H^k_{\text{APS},t}(SM|_\Sigma))} \text{, is continuous} . \tag{4.10}$$

This follows from the fact that $\widetilde{D}_\bullet$, $\widetilde{A}_\bullet$ are families of operators whose coefficients depend smoothly on t, moreover, the spectrum of $\widetilde{D}_t^{\mathrm{APS}}$ depends continuously in t as well [49, Chap. II, §5].

With this observation let $\tau \in \mathbb{R}$ and let T be the linear operator on $C(I_\tau, H^k_{\mathrm{APS},\bullet}(SM|_\Sigma))$ defined by the integral equation to (4.9)

$$(Tu)_t := U\psi_0 + \int_0^t [\widetilde{f}_s - iJ_s^{(\varepsilon)}\widetilde{D}_s^{\mathrm{APS}}J_s^{(\varepsilon)}u_s]\mathrm{d}s\,,$$

where $I_\tau = (0,\tau)$ is a short notation – notice that the integral is well-defined because $J_s^{(\varepsilon)}\widetilde{D}_s^{\mathrm{APS}}J_s^{(\varepsilon)}u_s$ is continuous in s. Notice that by construction $Tu \in C^1(I_\tau, H^k_{\mathrm{APS},\bullet}(SM|_\Sigma))$, moreover, $C(I_\tau, H^k_{\mathrm{APS},\bullet}(SM|_\Sigma))$ is a Banach space with norm

$$\|u\|_{C(I_\tau, H^k_{\mathrm{APS},\bullet}(SM|_\Sigma))} := \sup_{t\in(0,\tau)} \|u_t\|_{H^k_{\mathrm{APS},t}(SM|_\Sigma)}\,.$$

Finally, provided τ is small enough, T is a contraction as

$$\|Tu - Tv\|_{C(I_\tau, H^k_{\mathrm{APS},\bullet}(SM|_\Sigma))} \leqslant |\tau| \sup_{t\in(0,\tau)} \|J_t^{(\varepsilon)}\widetilde{D}_t^{\mathrm{APS}}J_t^{(\varepsilon)}\|_{B(H^k_{\mathrm{APS},t}(SM|_\Sigma))}\|u-v\|_{C(I_\tau, H^k_{\mathrm{APS},\bullet}(SM|_\Sigma))}\,,$$

where the supremum of $\|J_t^{(\varepsilon)}\widetilde{D}_t^{\mathrm{APS}}J_t^{(\varepsilon)}\|_{B(H^k_{\mathrm{APS},t}(SM|_\Sigma))}$ is finite because $J_t^{(\varepsilon)}\widetilde{D}_t^{\mathrm{APS}}J_t^{(\varepsilon)}$ depends at least continuously in t. Therefore, for small enough $\tau \in \mathbb{R}$ and for all $k \in \mathbb{N}$, the Banach-Caccioppoli fixed-point Theorem entails the existence and uniqueness of $\widetilde{\psi}^{(\varepsilon)} \in C(I_\tau, H^k_{\mathrm{APS},\bullet}(SM|_\Sigma))$, in particular $\widetilde{\psi}^{(\varepsilon)} = T\widetilde{\psi}^{(\varepsilon)} \in C^1(I_\tau, H^k_{\mathrm{APS},\bullet}(SM|_\Sigma))$. The existence of a global solution follows from the fact that (4.9) is a linear ODE.

Finally the uniqueness of the solution $\widetilde{\psi}^{(\varepsilon)}$ and the inclusion $H^{k+1}_{\mathrm{APS},t}(SM|_\Sigma) \subset H^k_{\mathrm{APS},t}(SM|_\Sigma)$ entails that, in fact, $\widetilde{\psi}^{(\varepsilon)} \in C^1(\mathbb{R}, H^\infty_{\mathrm{APS},\bullet}(SM|_\Sigma))$.

2. *Convergent subsequence:* We shall now prove that $\widetilde{\psi}^{(\varepsilon)}$ admits a subsequence $\{\widetilde{\psi}^{(\varepsilon_j)}\}_j$ which converges locally in $C(\mathbb{R}, H^\infty_{\mathrm{APS},\bullet}(SM|_\Sigma))$ – *i.e.* it converges in $C(I, H^\infty_{\mathrm{APS},\bullet}(SM|_\Sigma))$ for all compact interval $I \subset \mathbb{R}$. This will be a consequence of the relative compactness of $\widetilde{\Psi} := \{\widetilde{\psi}^{(\varepsilon)} \,|\, \varepsilon > 0\}$ in $C(I, H^k(SM|_\Sigma))$ for all $k \in \mathbb{N}$, a fact which we will prove using the Arzelà-Ascoli theorem.

   2.a *Proof that $\widetilde{\Psi}_t$ is relatively compact:* Let $I \subset \mathbb{R}$ be a compact interval and let $t \in I$ be arbitrary but fixed: We shall now prove that $\widetilde{\Psi}_t := \{\widetilde{\psi}_t^{(\varepsilon)} \,|\, \varepsilon > 0\} \subset H^k(SM|_\Sigma)$ is relatively compact for all $k \in \mathbb{Z}$. To this avail we shall prove that, for all $k \in \mathbb{N}$, $\widetilde{\Psi}_t$ is bounded in $H^{2k}(SM|_\Sigma)$: the compactness of the inclusion $H^{2k}(SM|_\Sigma) \hookrightarrow H^{2k-1}(SM|_\Sigma)$ will then imply that $\widetilde{\Psi}_t$ is relatively compact in $H^{2k-1}(SM|_\Sigma)$ for all $k \in \mathbb{N}$. To prove that $\widetilde{\Psi}$ is bounded in $H^{2k}(SM|_\Sigma)$ we shall estimate $\|\widetilde{\psi}_t^{(\varepsilon)}\|_{H^{2k}(SM|_\Sigma)}$ by an ε-independent constant.

   A direct inspection leads to

$$\begin{aligned}\partial_t\|\widetilde{\psi}_t^{(\varepsilon)}\|^2_{H^{2k}(SM|_\Sigma)} &= \partial_t(\langle\widetilde{D}_t\rangle^{2k}\widetilde{\psi}_t^{(\varepsilon)}, \langle\widetilde{D}_t\rangle^{2k}\widetilde{\psi}_t^{(\varepsilon)})_{L^2(SM|_\Sigma)}\\ &= 2\,\mathrm{Re}([\partial_t, \langle\widetilde{D}_t\rangle^{2k}]\widetilde{\psi}_t^{(\varepsilon)}, \langle\widetilde{D}_t\rangle^{2k}\widetilde{\psi}_t^{(\varepsilon)})_{L^2(SM|_\Sigma)}\\ &\quad + 2\,\mathrm{Re}(\langle\widetilde{D}_t\rangle^{2k}\partial_t\widetilde{\psi}_t^{(\varepsilon)}, \langle\widetilde{D}_t\rangle^{2k}\widetilde{\psi}(\varepsilon)_t)_{L^2(SM|_\Sigma)}\,,\end{aligned}$$

   where $\langle\widetilde{D}_t\rangle^{2k} = (1+\widetilde{D}_t^2)^k$.

   The term $[\partial_t, \langle\widetilde{D}_t\rangle^{2k}]$ is a differential operator of order $2k$ on $\Gamma(SM|_\Sigma)$ – notice that ∂_t only acts on the coefficients of $\langle\widetilde{D}_t\rangle^{2k}$. Therefore, it leads to a continuous map

$$[\partial_t, \langle\widetilde{D}_t\rangle^{2k}] \colon H^{\ell+2k}(SM|_\Sigma) \to H^\ell(SM|_\Sigma)\,.$$

It follows that the first term in the latter equation is bounded by

$$2\,\mathrm{Re}([\partial_t, \langle \widetilde{D}_t \rangle^{2k}]\widetilde{\psi}_t^{(\varepsilon)}, \langle \widetilde{D}_t \rangle^{2k}\widetilde{\psi}_t^{(\varepsilon)})_{L^2(SM|_\Sigma)} \leqslant c_1(t)\|\widetilde{\psi}_t^{(\varepsilon)}\|^2_{H^{2k}(SM|_\Sigma)}\,,$$

where $c_1(t)$ depends smoothly on t and it is ε-independent. Together with (4.6) we have

$$\begin{aligned}\partial_t\|\widetilde{\psi}_t^{(\varepsilon)}\|^2_{H^{2k}(SM|_\Sigma)} \leqslant\ & c_1(t)\|\widetilde{\psi}_t^{(\varepsilon)}\|^2_{H^{2k}(SM|_\Sigma)} \\ & + 2i\,\mathrm{Im}(\langle \widetilde{D}_t^{\mathrm{APS}} \rangle^{2k} J_t^{(\varepsilon)}\widetilde{D}_t^{\mathrm{APS}} J_t^{(\varepsilon)}\widetilde{\psi}_t^{(\varepsilon)}, \langle \widetilde{D}_t^{\mathrm{APS}} \rangle^{2k}\widetilde{\psi}_t^{(\varepsilon)})_{L^2(SM|_\Sigma)} \\ & + 2\,\mathrm{Re}(\langle \widetilde{D}_t^{\mathrm{APS}} \rangle^{2k}\widetilde{f}_t, \langle \widetilde{D}_t^{\mathrm{APS}} \rangle^{2k}\widetilde{\psi}_t^{(\varepsilon)})_{L^2(SM|_\Sigma)}\,, \\ \leqslant\ & c_2(t)\|\widetilde{\psi}_t^{(\varepsilon)}\|^2_{H^{2k}(SM|_\Sigma)} + \|\widetilde{f}_t\|^2_{H^{2k}(SM|_\Sigma)}\,,\end{aligned}$$

where $c_2 = c_1 + 1$ while in the last equality we exploited the self-adjointness of the operator $J_t^{(\varepsilon)}\widetilde{D}_t^{\mathrm{APS}} J_t^{(\varepsilon)}$ as well as the bound

$$2\,\mathrm{Re}(\langle \widetilde{D}_t^{\mathrm{APS}} \rangle^{2k}\widetilde{f}_t, \langle \widetilde{D}_t^{\mathrm{APS}} \rangle^{2k}\widetilde{\psi}_t^{(\varepsilon)})_{L^2(SM|_\Sigma)} \leqslant \|\widetilde{f}_t\|^2_{H^{2k}(SM|_\Sigma)} + \|\widetilde{\psi}_t^{(\varepsilon)}\|^2_{H^{2k}(SM|_\Sigma)}\,.$$

Integration over t leads to

$$\|\widetilde{\psi}_t^{(\varepsilon)}\|^2_{H^{2k}(SM|_\Sigma)} \leqslant \|U\psi_0\|^2_{H^{2k}(SM|_\Sigma)} + \int_0^t \|\widetilde{f}_s\|^2_{H^{2k}(SM|_\Sigma)}\mathrm{d}s + \int_0^t c_2(s)\|\psi_s^{(\varepsilon)}\|^2_{H^{2k}(SM|_\Sigma)}\mathrm{d}s\,.$$

It follows by Grönwall's Lemma that

$$\|\widetilde{\psi}_t^{(\varepsilon)}\|^2_{H^{2k}(SM|_\Sigma)} \leqslant \left(\|U\psi_0\|^2_{H^{2k}(SM|_\Sigma)} + \int_0^t \|\widetilde{f}_s\|^2_{H^{2k}(SM|_\Sigma)}\mathrm{d}s\right)\exp\left[\int_{t_0}^t c_2(s)\mathrm{d}s\right]\,,$$

and therefore $\widetilde{\Psi}_t := \{\widetilde{\psi}_t^{(\varepsilon)}\,|\,\varepsilon > 0\}$ is bounded in $H^{2k}(SM|_\Sigma)$. Since $H^{2k}(SM|_\Sigma) \hookrightarrow H^{2k-1}(SM|_\Sigma)$ is compact it follows that $\widetilde{\Psi}_t$ is relatively compact on $H^{2k-1}(SM|_\Sigma)$. Since this holds for all $k \geqslant 1$, we have found that $\widetilde{\Psi}_t$ is relatively compact in $H^k(SM|_\Sigma)$ for all $k \in \mathbb{N}$.

2.b *Proof that $\widetilde{\Psi}$ is equicontinuous:* In order to apply Arzelà-Ascoli Theorem, we shall now prove that the family $\widetilde{\Psi} := \{\widetilde{\psi}^{(\varepsilon)}\,|\,\varepsilon > 0\} \subset C(I, H^k(SM|_\Sigma))$, $I \subset \mathbb{R}$ compact interval, is equicontinuous for all $k \in \mathbb{N}$. For that we shall consider

$$\begin{aligned}\|\partial_t\widetilde{\psi}_t^{(\varepsilon)}\|_{H^k_{\mathrm{APS},t}(SM|_\Sigma)} &= \|\widetilde{f}_t - iJ_t^{(\varepsilon)}\widetilde{D}_t^{\mathrm{APS}} J_t^{(\varepsilon)}\widetilde{\psi}_t^{(\varepsilon)}\|_{H^k_{\mathrm{APS},t}(SM|_\Sigma)} \\ &\leqslant \|\widetilde{f}_t\|_{H^k_{\mathrm{APS},t}(SM|_\Sigma)} + \|\widetilde{\psi}_t^{(\varepsilon)}\|_{H^{k+1}_{\mathrm{APS},t}(SM|_\Sigma)} \leqslant C_I\,,\end{aligned}$$

where $C_I > 0$ does not depend on ε and t?. Here we used that $\|J_t^{(\varepsilon)}\|_{B(H^\bullet_{\mathrm{APS},t}(SM|_\Sigma))} \leqslant 1$ as well as $z \leqslant \langle z \rangle = \sqrt{1+|z|^2}$ for all $z \in \mathbb{R}$. It then follows that for all $t, s \in I$,

$$\begin{aligned}\|\widetilde{\psi}_t^{(\varepsilon)} - \widetilde{\psi}_s^{(\varepsilon)}\|_{H^k(SM|_\Sigma)} &\leqslant \int_s^t \|\partial_\tau\widetilde{\psi}_\tau^{(\varepsilon)}\|_{H^k(SM|_\Sigma)}\mathrm{d}\tau \\ &\leqslant \int_s^t c(\tau)\|\partial_\tau\widetilde{\psi}_\tau^{(\varepsilon)}\|_{H^k_{\mathrm{APS},\tau}(SM|_\Sigma)}\mathrm{d}\tau \\ &\leqslant |t-s|\sup_{\tau\in I}|c(\tau)|\|\widetilde{\psi}_\tau^{(\varepsilon)}\|_{H^{k+1}_{\mathrm{APS},\tau}(SM|_\Sigma,\mu_\Sigma^{N_\tau})} \\ &\leqslant |t-s|\widetilde{C}_I\,.\end{aligned}$$

where $\widetilde{C}_I > 0$ does not depend on ε. In the second inequality we used the fact that $\|\ \|_{H^k(SM|_\Sigma)}$ and $\|\ \|_{H^k_{\mathrm{APS},t}(SM|_\Sigma)}$ are equivalent for all t. Therefore $\widetilde{\Psi}$ is equicontinuous in $C^0(\mathbb{R}, H^k(SM|_\Sigma))$.

2.c *Application of Arzelà-Ascoli Theorem:*
Since $\widetilde{\Psi}$ is a bounded and equicontinuous family in $C(I, H^k(SM|_\Sigma))$ such that $\widetilde{\Psi}_t$ is relatively compact in $H^k(SM|_\Sigma)$ for all $t \in I$, the Arzelà-Ascoli theorem [50] ensures that $\widetilde{\Psi}$ is relatively compact in $C(I, H^k(SM|_\Sigma))$ – this holds for all $k \in \mathbb{N}$.
Therefore there exists a subsequence $\{\widetilde{\psi}^{(\varepsilon_j)}\}_{j\in\mathbb{N}}$ of $\{\widetilde{\psi}^{(\varepsilon)}\}$ which converges to $\widetilde{\psi} \in C(I, H^k(SM|_\Sigma))$ as $j \to +\infty$. This entails in particular that $\widetilde{\psi}_t^{(\varepsilon_j)}$ converges in $H^k(SM|_\Sigma)$ for all $t \in I$. It follows that, $\widetilde{\psi}_t^{(\varepsilon)} = \lim_{j\to+\infty} \widetilde{\psi}_t^{(\varepsilon_j)} \in H^k_{\mathrm{APS},t}(SM|_\Sigma)$ because $H^k_{\mathrm{APS},t}(SM|_\Sigma)$ is closed in $H^k(SM|_\Sigma)$. This implies that $\widetilde{\psi} \in C(I, H^k_{\mathrm{APS},\bullet}(SM|_\Sigma))$.
Finally, using a diagonal subsequence argument we can assume that $\widetilde{\psi}^{(\varepsilon_j)}$ converges to $\widetilde{\psi}$ locally uniformly in $C(\mathbb{R}, H^k_{\mathrm{APS},\bullet}(SM|_\Sigma))$ for all $k \geqslant 0$. Since this holds for all $k \in \mathbb{N}$, we have $\widetilde{\psi} \in C(\mathbb{R}, H^\infty_{\mathrm{APS},\bullet}(SM|_\Sigma))$. In particular $\widetilde{\psi}_t \in H^\infty(SM|_\Sigma)$ for all $t \in \mathbb{R}$.

3. *Equation of motion:* We shall now prove that $\widetilde{\psi}$ solves the Cauchy problem (4.6). For all $j \in \mathbb{N}$ we have

$$\widetilde{\psi}_t^{(\varepsilon_j)} - U\psi_0 - \int_0^t \widetilde{f}_s \mathrm{d}s = -i \int_0^t J_s^{(\varepsilon_j)} \widetilde{D}_s J_s^{(\varepsilon_j)} \widetilde{\psi}_s^{(\varepsilon_j)} \mathrm{d}s\,.$$

The left-hand side of this equation converges to $\widetilde{\psi}_t - U\psi_0 - \int_0^t \widetilde{f}_s \mathrm{d}s$ in $H^k(SM|_\Sigma)$ for all $k \in \mathbb{N}$. For all such k the norm of the right-hand side can be estimated as

$$\| \int_0^t J_s^{(\varepsilon_j)} \widetilde{D}_s J_s^{(\varepsilon_j)} \widetilde{\psi}_s^{(\varepsilon_j)} \mathrm{d}s \|_{H^k(SM|_\Sigma)} \leqslant \int_0^t c(s) \| J_s^{(\varepsilon_j)} \widetilde{D}_s J_s^{(\varepsilon_j)} \widetilde{\psi}_s^{(\varepsilon_j)} \|_{H^k_{\mathrm{APS},s}(SM|_\Sigma)} \mathrm{d}s\,,$$

where we exploited that $\| \ \|_{H^k(SM|_\Sigma)}$ and $\| \ \|_{H^k_{\mathrm{APS},s}(SM|_\Sigma)}$ are equivalent on $H^k_{\mathrm{APS},s}(SM|_\Sigma)$. Moreover, notice that the integrand is a continuous function of s which pointwise converges to $\|\widetilde{D}_s \widetilde{\psi}_s\|_{H^k_{\mathrm{APS},s}(SM|_\Sigma)}$ as $j \to +\infty$. In fact, this follows from the estimate

$$\begin{aligned}
\|J_s^{(\varepsilon_j)} \widetilde{D}_s J_s^{(\varepsilon_j)} \widetilde{\psi}_s^{(\varepsilon_j)} - \widetilde{D}_s \widetilde{\psi}_s\|_{H^k_{\mathrm{APS},s}(SM|_\Sigma)} &\leqslant \|J_s^{(\varepsilon_j)} \widetilde{D}_s J_s^{(\varepsilon_j)} (\widetilde{\psi}_s^{(\varepsilon_j)} - \widetilde{\psi}_s)\|_{H^k_{\mathrm{APS},s}(SM|_\Sigma)} \\
&\quad + \|J_s^{(\varepsilon_j)} \widetilde{D}_s (J_s^{(\varepsilon_j)} - 1) \widetilde{\psi}_s\|_{H^k_{\mathrm{APS},s}(SM|_\Sigma)} + \|(J_s^{(\varepsilon_j)} - 1) \widetilde{D}_s \widetilde{\psi}_s\|_{H^k_{\mathrm{APS},s}(SM|_\Sigma)} \\
&\leqslant \|\widetilde{\psi}_s^{(\varepsilon_j)} - \widetilde{\psi}_s\|_{H^{k+1}_{\mathrm{APS},s}(SM|_\Sigma)} + 2\|(J_s^{(\varepsilon_j)} - 1)\widetilde{\psi}_s\|_{H^{k+1}_{\mathrm{APS},s}(SM|_\Sigma)}\,,
\end{aligned}$$

and therefore the claims follows from the convergence $\widetilde{\psi}^{(\varepsilon_j)} \to \widetilde{\psi}$ as well as the strong convergence of $J_s^{(\varepsilon_j)}$ to the identity operator in $H^\bullet_{\mathrm{APS},s}(SM|_\Sigma)$.
A dominated convergence arguments leads to

$$\widetilde{\psi}_t - U\psi_0 - \int_0^t \widetilde{f}_s \mathrm{d}s = -i \int_0^t \widetilde{D}_s \widetilde{\psi}_s \mathrm{d}s\,,$$

which equivalent to the fulfillment of the reduced Dirac equation (4.3) together with the initial value $\widetilde{\psi}_0 = U\psi_0$.
Finally we observe that $\widetilde{\psi} \in C^1(\mathbb{R}, H^\infty(SM|_\Sigma))$ since $\partial_t \widetilde{\psi}_t = \widetilde{f}_t - i\widetilde{D}_t \widetilde{\psi}_t \in H^\infty(SM|_\Sigma)$, moreover,

$$\partial_t^2 \widetilde{\psi}_t = \partial_t \widetilde{f}_t - i[\widetilde{D}_t, \partial_t]\widetilde{\psi}_t - i\widetilde{D}_t \widetilde{f}_t - \widetilde{D}_t^2 \widetilde{\psi}_t\,.$$

Since the right-hand side of the latter equation lies in $H^\infty(SM|_\Sigma)$ we find that $\widetilde{\psi} \in C^2(\mathbb{R}, H^\infty(SM|_\Sigma))$. An induction argument shows that $\widetilde{\psi} \in C^\ell(\mathbb{R}, H^\infty(SM|_\Sigma))$ for all $\ell \in \mathbb{N}$, that is, $\widetilde{\psi} \in C^\infty(\mathbb{R}, H^\infty(SM|_\Sigma))$. We thus have $\widetilde{\psi} \in C^\infty(\mathbb{R}, \Gamma_{\mathrm{APS}}(SM|_\Sigma))$.

This shows the existence of smooth solutions.

Remark 4.2. The proof presented above does not really depend on the actual form of APS boundary conditions. In fact, the proof of Theorem 1.2 (under Assumption 4) is still valid provided we consider any family $B_\bullet = \{B_t\}_{t\in\mathbb{R}}$ of elliptic self-adjoint boundary conditions [6, Definition 3.5-3.7] for $\widetilde{D}_\bullet$ such that (4.10) holds true.

This observation applies in particular in the following setting (which we shall consider in a moment). Suppose $\partial\Sigma$ consists of two connected component $\partial\Sigma^{(1)}$, $\partial\Sigma^{(2)}$ and let $\partial M = \partial M^{(1)} \cup \partial M^{(2)}$ the associated decomposition of ∂M. In this situation we may consider the Cauchy problem for the Dirac operator with APS boundary condition on $\partial M^{(1)}$ and MIT boundary condition on $\partial M^{(2)}$. The latter boundary condition has been investigated in [47] and consist in requiring that $\psi|_{\partial M^{(2)}}$ satisfies

$$(\mathrm{Id} + i\gamma_M(e_n))\psi|_{\partial M^{(2)}} = 0\,. \tag{4.11}$$

Notice in particular that MIT boundary conditions are local and depend smoothly on time, therefore, condition (4.10) is fulfilled and the proof of Theorem 1.2 (within assumption 4) still holds true. Moreover, since MIT boundary conditions are local, the solution ψ to the Cauchy problem shares better propagation properties. In fact the bound (3.4) can be improved to

$$\mathrm{supp}(\psi) \subseteq J(\mathrm{supp}(f) \cup \mathrm{supp}(\psi_0)) \cup J(\partial\Sigma_0^{(1)})\,, \tag{4.12}$$

where notice that no contribution from $J(\partial\Sigma_0^{(2)})$ arises on the right-hand side.

4.3 Dropping Assumption 4

We shall now discuss the case when Σ is non-compact – notice that we are still assuming that $\partial\Sigma$ is compact, *cf.* Assumption 1. To this avail we shall follow [5] – see also [46, Proposition 3.4]. On account of Proposition 3.2 any solution ψ to the Cauchy problem (1.1) satisfies

$$\mathrm{supp}(\psi) \subseteq J(\mathrm{supp}(\psi_0)) \cup J(\mathrm{supp}(f)) \cup J(\partial\Sigma_0)\,.$$

Let $K := \mathrm{supp}(\psi_0) \cup \mathrm{supp}(f)$ and let $T > 0$ be large enough to that $K \subseteq (-T, T) \times \Sigma =: M_T$ – Notice that M_T is again a globally hyperbolic spacetime with timelike boundary.

Let $\widetilde{\Sigma}$ be the projection on Σ_0 of $(J(K) \cup J(\Sigma_0)) \cap M_T$ with respect to $M_T = T \times \Sigma \to \Sigma$. Let us consider $\widetilde{U}$ a relatively compact subset of Σ with smooth boundary and such that $\widetilde{\Sigma} \subset \widetilde{U}$ – notice that the closure of $\widetilde{\Sigma}$ is compact. Notice that by construction $\partial\Sigma \subset \widetilde{\Sigma} \subset \widetilde{U}$.

Let $V := \partial\widetilde{U}\backslash\partial\Sigma$ and consider a smooth change of the metric h_t so that h_t becomes a product metric in a neighborhood of V – notice that $\mathrm{dist}(V, \partial\Sigma) > 0$ since $\partial\Sigma$ is closed. Such change can be realized smoothly in t and in a such a way that it does not affect the metric in $\widetilde{\Sigma}$.

We thus consider the doubling $\widehat{U}$ of $\widetilde{U}$ along V. The resulting manifold $\widehat{U}_{(-T,T)} := (-T, T) \times \widehat{U}$ is globally hyperbolic with timelike boundary and compact Cauchy surface $\widehat{U}$. However, the boundary $\partial\widehat{U}$ of $\widehat{U}$ is made of two disconnected component $\partial\widehat{U}^{(1)}$, $\partial\widehat{U}^{(2)}$ each of which is a copy of $\partial\Sigma$. Therefore for each t the APS boundary conditions do not suffice to make the $\widetilde{D}_t$ self-adjoint on $\widehat{U}$, as no boundary conditions have been imposed on the second copy of $\partial\Sigma$.

To cope with this problem we shall profit of Remark 4.2 and consider the Cauchy problem for the Dirac operator on $\widehat{U}_{(-T,T)}$ with boundary conditions given by APS boundary conditions on $\partial\widehat{U}^{(1)}_{(-T,T)}$ and MIT boundary conditions on $\partial\widehat{U}^{(2)}_{(-T,T)}$. Since $\mathrm{supp}(\psi_0) \subseteq \widetilde{U} \subset \widehat{U}$ and $\mathrm{supp}(f) \subset \widetilde{U}_{(-T,T)} \subset \widehat{U}_{(-T,T)}$, it follows that ψ_0 and f can be regarded as data for the Cauchy problem (1.1) on $\widehat{U}_{(-T,T)}$. Since $\widehat{U}$ is compact, we are reduced to the proof of Theorem 1.2 under Assumption 4. On account of Remark 4.2 such Cauchy problem has a unique solution $\psi_T \in \Gamma_{\mathrm{BC}}(S\widehat{U}_{(-T,T)})$ – here $_{\mathrm{BC}}$ is a short notation for the aforementioned boundary conditions. Moreover, the support of ψ_T fulfills

$$\mathrm{supp}(\psi_T) \subseteq J(\mathrm{supp}(\psi_0) \cup \mathrm{supp}(f)) \cup J(\partial\widehat{U}^{(1)}) \subseteq M_T\,,$$

where $J(\partial\widehat{U}^{(1)}) = J(\partial\Sigma)$ (*cf.* Equation (4.12)). It follows that, in fact, $\psi_T \in \Gamma_{\text{APS}}(SM|_{M_T})$. Finally, Proposition 3.3 implies that $\psi_T = \psi_{T'}$ for all $T' > T$, therefore, we find $\psi \in \Gamma_{\text{APS}}(SM)$ such that $\psi|_{M_T} = \psi_T$ for all $T \in \mathbb{R}$.

5 Well-posedness of the Cauchy problem

In this section we put everything together to prove Theorem 1.2. For that we still need to define the involved topologies (compare [4, Section 2]):

The space of smooth sections $\Gamma(.)$ on a smooth vector bundle together with the standard family of seminorms is a Fréchet topological vector space. We equip $\Gamma_K(SM|_\Sigma) := \{\psi \in \Gamma_K(SM|_\Sigma) \mid \text{supp}(\psi) \subset K\}$ for a closed subset $K \subset \Sigma$ with the subspace topology. Then, $\Gamma_{\text{APS}}(SM)$ resp. $\Gamma_{\text{cc}}(SM|_\Sigma)$ is equipped respectively with the relative locally convex topology of $\Gamma(SM)$ and with the inductive topology of the system $\{\Gamma_K(SM|_\Sigma)\}_{K\in\mathsf{K}_{\partial\Sigma}}$ – here $\mathsf{K}_{\partial\Sigma}$ contains compact subsets $K \subseteq \Sigma$ with $K \cap \partial\Sigma = \varnothing$. Similarly $\Gamma_{\text{cc}}(SM)$ is equipped with the inductive topology of the system $\{\Gamma_K(SM)\}_{K\in\mathsf{K}_{\partial M}}$.

Proof of Theorem 1.2. On account of Section 4.2.3 and 4.3, we have seen that there exists a smooth solution of the Cauchy problem 1.1. Furthermore, on account of Proposition 3.3 it follows that the solution is unique. To conclude our proof, it remain to show that the solution depends continuously on the Cauchy data. For that we consider the linear map

$$\mathsf{D}_M^{\text{APS}} \colon \Gamma_{\text{APS}}(SM) \to \Gamma(SM|_\Sigma) \oplus \Gamma(SM)\,, \qquad \psi \mapsto \psi|_{\Sigma_0} \oplus D_M\psi\,,$$

which is continuous. Let $K_1 \in \mathsf{K}_{\partial\Sigma}$ and $K_2 \in \mathsf{K}_{\partial M}$. Then $\Gamma_{K_1}(SM|_\Sigma) \oplus \Gamma_{K_2}(SM)$ is closed (and therefore Fréchet) and so is $V_{K_1,K_2} := (\mathsf{D}^{\text{APS}})^{-1}(\Gamma_{K_1}(SM|_\Sigma) \oplus \Gamma_{K_2}(SM)) \subset \Gamma_{\text{APS}}(SM)$. Moreover, the previous part of the proof shows that $\mathsf{D}^{\text{APS}} \colon V_{K_1,K_2} \to \Gamma_{K_1}(SM|_\Sigma) \oplus \Gamma_{K_2}(SM)$ is a bijection. The open mapping theorem for Fréchet spaces entails that

$$(\mathsf{D}^{\text{APS}}|_{V_{K_1,K_2}})^{-1} \colon V_{K_1,K_2} \to \Gamma_{\text{APS}}(SM)$$

is continuous. The claim then follows from the arbitrariness of K_1 and K_2. □

A byproduct of the well-posedness of the Cauchy problem is the existence of Green operators with similar properties to the ones found in [28, 29].

Proposition 5.1. *The classical Dirac operator on a globally hyperbolic manifold with timelike boundary coupled with APS boundary conditions is Green-hyperbolic, i.e., there exist linear maps* $G^\pm_{\text{APS}} \colon \Gamma_{\text{cc}}(SM) \to \Gamma_{\text{APS}}(SM)$ *satisfying*

$$D_M G^\pm_{\text{APS}} f = f\,, \qquad G^\pm_{\text{APS}} D_M f = f\,. \tag{5.1}$$

for all $f \in \Gamma_{\text{cc}}(SM)$.

Proof. Let $f \in \Gamma_{\text{cc}}(SM)$ and choose $t_0 \in \mathbb{R}$ such that $\text{supp}(f) \subset J^+(\Sigma_{t_0})$. By Theorem 1.2, there exists a unique solution $\psi(f) \in \Gamma_{\text{APS}}(SM)$ to the Cauchy problem

$$D_M\psi = f\,, \qquad \psi|_{\Sigma_{t_0}} = 0\,, \qquad \psi \in \Gamma_{\text{APS}}(SM)\,.$$

Setting $G^+_{\text{APS}} f := \psi_f$ leads to the wanted operator. The existence of G^-_{APS} is proven analogously. □

References

[1] L. Aké, J. L. Flores, M. Sánchez, *Structure of globally hyperbolic spacetimes with timelike boundary.* to appear in Rev. Mat. Iberoamericana (2020).

[2] M. F. Atiyah, I. M. Singer, *The index of elliptic operators I*, Ann. of Math. **87**, 484-530 (1968).

[3] M. F. Atiyah, V. K. Patodi, I. M. Singer, *Spectral asymmetry and Riemannian geometry*, I. Math. Proc. Cambridge Philos. Soc. **77**, 43-69 (1975).

[4] C. Bär, *Green-hyperbolic operators on globally hyperbolic spacetimes*, Comm. Math. Phys. **333**, 1585 (2015).

[5] C. Bär, *Geometric Wave Equations*, Lecture notes, 2016. https://www.math.uni-potsdam.de/fileadmin/user_upload/Prof-Geometrie/Dokumente/Lehre/Lehrmaterialien/Waves.pdf

[6] C. Bär, W. Ballmann, *Guide to Elliptic Boundary Value Problems for Dirac-Type Operators.* (2016) In: Ballmann W., Blohmann C., Faltings G., Teichner P., Zagier D. (eds) Arbeitstagung Bonn 2013. Progress in Mathematics, vol 319. Birkhäuser, Cham.

[7] C. Bär, P. Gauduchon, A. Moroianu, *Generalized cylinders in semi-Riemannian and spin geometry.* Math. Zeit. **249**, 545-580 (2005).

[8] C. Bär, N. Ginoux, *Classical and quantum fields on Lorentzian manifolds.* in: C. Bär, J. Lohkamp and M. Schwarz (eds.), Global Differential Geometry, 359-400, Springer-Verlag Berlin Heidelberg (2012).

[9] C. Bär, N. Ginoux, *CCR- versus CAR-Quantization on Curved Spacetimes* in: F. Finster, O. Müller, M. Nardmann, J. Tolksdorf and E. Zeidler (eds.), Quantum Field Theory and Gravity, 183-206, Springer Basel AG (2012).

[10] C. Bär, A. Strohmaier, *A Rigorous Geometric Derivation of the Chiral Anomaly in Curved Backgrounds*, Comm. Math. Phys. **347**, 703-721 (2016).

[11] C. Bär, A. Strohmaier, *An index theorem for Lorentzian manifolds with compact spacelike Cauchy boundary*, Amer. J. Math. **141**, 1412-1455 (2019).

[12] C. Bär, A. Strohmaier, *Local Index Theory for Lorentzian Manifolds*, arXiv:2012.01364

[13] A. N. Bernal, M. Sánchez, *On Smooth Cauchy Hypersurfaces and Geroch?s Splitting Theorem.* Comm. Math. Phys. **243**, 461-470 (2003).

[14] A. N. Bernal, M. Sanchez, *Smoothness of time functions and the metric splitting of globally hyperbolic spacetimes*, Comm. Math. Phys. **257**, 43 (2005).

[15] A. N. Bernal, M. Sanchez, *Further results on the smoothability of Cauchy hypersurfaces and Cauchy time functions*, Lett. Math. Phys. **77**, 183 (2006).

[16] H. Baum, *Spin-Strukturen und Dirac-Operatoren über pseudoriemannschen Mannigfaltigkeiten.* B. G. Teubner Verlagsgesellschaft, Leipzig, (1981).

[17] M. Benini, C. Dappiaggi, *Models of free quantum field theories on curved backgrounds,* in: R. Brunetti, C. Dappiaggi, K. Fredenhagen and J. Yngvason (eds.), Advances in Algebraic Quantum Field Theory, 75-124, Springer-Verlag, Heidelberg (2015).

[18] M. Benini, C. Dappiaggi, A. Schenkel, *Algebraic quantum field theory on spacetimes with timelike boundary.* Annales Henri Poincaré **19**, 2401-2433 (2018).

[19] M. Benini, M. Perin, A. Schenkel, *Model-Independent Comparison Between Factorization Algebras and Algebraic Quantum Field Theory on Lorentzian Manifolds.* Comm. Math. Phys. **377**, 971-997(2020).

[20] M. Benini, A. Schenkel, *Quantum Field Theories on Categories Fibered in Groupoids.* Comm. Math. Phys. **356**, 19-64 (2017).

[21] M. Benini, A. Schenkel, L. Woike, *Homotopy theory of algebraic quantum field theories.* Lett. Math. Phys. **109**, 1487-1532 (2019).

[22] R. Brunetti, C. Dappiaggi, K. Fredenhagen, J. Yngvason, *Advances in algebraic quantum field theory.* Springer (2015)

[23] R. Brunetti, K. Fredenhagen, N. Pinamonti, *Algebraic approach to Bose Einstein Condensation in relativistic Quantum Field Theory. Spontaneous symmetry breaking and the Goldstone Theorem,* arXiv:1911.01829 (2019).

[24] D. Buchholz, K. Fredenhagen, *A C^*-algebraic approach to interacting quantum field theories.* Comm. Math. Phys. **377**, 947-969 (2020).

[25] M. Capoferri, C. Dappiaggi, N. Drago, *Global wave parametrices on globally hyperbolic spacetimes.* J. Math. Anal. App. **490**, 124316 (2020).

[26] M. Capoferri, M. Levitin, D. Vassiliev, *Geometric wave propagator on Riemannian manifolds.* to appear in Comm. Anal. Geom. (2020).

[27] M. Capoferri, D. Vassiliev, *Global propagator for the massless Dirac operator and spectral asymptotics.* arXiv:2004.06351 [math.AP] (2020).

[28] C. Dappiaggi, N. Drago, H. Ferreira, *Fundamental solutions for the wave operator on static Lorentzian manifolds with timelike boundary*, Lett Math Phys **109**, 2157 (2019).

[29] C. Dappiaggi, N. Drago, R. Longhi, *On Maxwell's Equations on Globally Hyperbolic Spacetimes with Timelike Boundary*, Ann. Henri Poincaré **21**, 2367-2409 (2020).

[30] C. Dappiaggi, F. Finster, S. Murro, E. Radici, *The Fermionic Signature Operator in De Sitter Spacetime,* J. Math. Anal. Appl. **485**, 123808 (2020).

[31] C. Dappiaggi, T.-P. Hack, N. Pinamonti *The extended algebra of observables for Dirac fields and the trace anomaly of their stress-energy tensor,* Rev. Math. Phys. **21**, 1241-1312 (2009).

[32] C. Dappiaggi, T.-P. Hack, K. Sanders, *Electromagnetism, local covariance, the Aharonov-Bohm effect and Gauss' law.* Comm. Math. Phys. **328**, 625 (2014).

[33] C. Dappiaggi, A. Marta, *A generalization of the propagation of singularities theorem on asymptotically anti-de Sitter spacetimes.* arXiv:2006.00560 (2020).

[34] C. Dappiaggi, V. Moretti, N. Pinamonti, *Rigorous construction and Hadamard property of the Unruh state in Schwarzschild spacetime.* Adv. Theor. Math. Phys. **15** 355 (2011)

[35] C. Dappiaggi, S. Murro, A. Schenkel, *Non-existence of natural states for Abelian Chern-Simons theory,* J. Geom. Phys. **116**, 119-123 (2017).

[36] C. Dappiaggi, G. Nosari, N. Pinamonti, *The Casimir effect from the point of view of algebraic quantum field theory.* Math. Phys. Anal. Geom. **19**, 12 (2016).

[37] P. A. M. Dirac, *The Quantum Theory of the Electron*, Proc. Roy. Soc. A, **117** (1938), 610-627.

[38] N. Drago, S. Murro, *A new class of Fermionic Projectors: Møller operators and mass oscillation properties,* Lett. Math. Phys. **107**, 2433-2451 (2017).

[39] M. Di Francesco, S. Fagioli and E. Radici, *Deterministic particle approximation for nonlocal transport equations with nonlinear mobility.* J. Differential Equations **266**, 2830-2868 (2019).

[40] S. Fagioli and E. Radici, *Solutions to aggregation-diffusion equations with nonlinear mobility constructed via a deterministic particle approximation.* Math. Models Methods Appl. Sci. **28** (9) 1801-1829 (2018).

[41] F. Finster, S. Murro, C. Röken, *The Fermionic Projector in a Time-Dependent External Potential: Mass Oscillation Property and Hadamard States,* J. Math. Phys. **57**, 072303 (2016).

[42] F. Finster, S. Murro, C. Röken, *The Fermionic Signature Operator and Quantum States in Rindler Space-Time,* J. Math. Anal. Appl. **454**, 385 (2017).

[43] K. Fredenhagen, K. Rejzner, *Quantum field theory on curved spacetimes: Axiomatic framework and examples,* J. Math. Phys. **57**, 031101 (2016).

[44] O. Gannot, M. Wrochna, *Propagation of singularities on AdS spacetimes for general boundary conditions and the holographic Hadamard condition.* to appear in J. Inst. Math. Jussieu.

[45] C. Gérard, *Microlocal Analysis of Quantum Fields on Curved Spacetimes.* ESI Lectures in Mathematics and Physics (2019).

[46] N. Ginoux, S. Murro, *On the Cauchy problem for Friedrichs systems on globally hyperbolic manifolds with timelike boundary.* arXiv:2007.02544 [math.AP] (2020).

[47] N. Große, S. Murro, *The well-posedness of the Cauchy problem for the Dirac operator on globally hyperbolic manifolds with timelike boundary.* Documenta Math. **25**, 737-765 (2020).

[48] T.-P. Hack, A. Schenkel, *Linear bosonic and fermionic quantum gauge theories on curved spacetimes.* Gen. Rel. Grav. **45**, 877 (2013)

[49] T. Kato, *Perturbation theory of linear operators*, Springer-Verlag Berlin Heidelberg (1995).

[50] R. E. Megginson, *An Introduction to Banach Space Theory* Graduate Texts in Mathematics 183, Springer-Verlag New York, Inc. October, 1998

[51] S. Murro, D. Volpe, *Intertwining operators for symmetric hyperbolic systems on globally hyperbolic manifolds.* Ann. Glob. Anal. Geom. 59, 1-25 (2021).

[52] V. Moretti, *Spectral theory and quantum mechanics. Mathematical foundations of quantum theories, symmetries and introduction to the algebraic formulation.* 2nd enlarged edition. Cham: Springer (2017)

[53] K. van den Dungen, *Families of spectral triples and foliations of space(time)*, J. Math. Phys. 59, 063507 (2018).

[54] A. Vasy, *The wave equation on asymptotically Anti-de Sitter spaces.* Analysis & PDE **5** 81 (2012).

[55] E. Witten, *Fermion Path Integrals And Topological Phases.* Rev. Mod. Phys. **88**, 35001 (2016).