Continuous and Analytic Dependence is an Unnecessary Requirement in Renormalization of Locally Covariant QFT

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Abstract

Finite renormalization freedom in locally covariant quantum field theories on curved spacetime is known to be tightly constrained, under certain standard hypotheses, to the same terms as in flat spacetime up to finitely many curvature dependent terms. These hypotheses include, in particular, locality, covariance, scaling and continuous and analytic dependence on the metric and coupling parameters. The analytic dependence hypothesis is somewhat unnatural, because it requires that locally covariant observables (which are simultaneously defined on all spacetimes) depend continuously on an arbitrary metric, with the dependence strengthened to analytic on analytic metrics. Moreover the fact that analytic metrics are globally rigid makes the implementation of this requirement at the level of local $*$-algebras of observables rather technically cumbersome. We show that the conditions of locality, covariance and scaling, in conjunction with the microlocal spectral condition, are actually sufficient to constrain the allowed finite renormalizations equally strongly, making both the continuity and the somewhat unnatural analyticity hypotheses unnecessary. The key step in the proof uses the non-linear Peetre theorem on the characterization of differential operators.

1 Introduction

Perturbative ultraviolet renormalization of locally covariant quantum field theories in (globally hyperbolic) curved spacetime is a well established topic of algebraic quantum field theory, especially for scalar fields [3, 4, 11, 12]. It essentially deals with two classes of objects: *Wick polynomials* and *time ordered Wick polynomials*. Exactly as in flat spacetime, these objects can be considered as the building blocks of the whole renormalization procedure. Smeared versions of Wick polynomials, of their time ordered products and of their derivatives generate an algebra $\mathcal{W}(M, \mathbf{g})$, for a given spacetime $(M, \mathbf{g})$, enlarged in a controlled way from the algebra of products of smeared linear fields. This enlarged

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algebra then includes physically fundamental observables, such as the stress-energy tensor, which is necessary, for instance, to evaluate the energy densities and fluxes of physical processes in curved spacetimes like particle creation or Hawking radiation. The stress-energy tensor is also needed to compute the back reaction of the quantum matter on the background geometry.

This paper deals only with Wick polynomials, though the presented results could in principle be adapted to deal also with their derivatives and their time ordered products. In curved spacetime, Wick polynomial have to satisfy stronger locality and covariance requirements than in flat spacetime. These requirements are conveniently stated in the language of category theory introduced in [5], which we also use here. We should stress, though, that the categorical language primarily serves to compress somewhat long lists of hypotheses into concise statements. Existence of locally covariant Wick polynomials and their time ordered products was established in the seminal works of Hollands and Wald, respectively in [11] and [12]. It is well known that, in flat spacetime, time ordered Wick polynomials are not uniquely defined. This fact survives the passage to curved spacetime. However, unlike in flat spacetime, the absence of a preferred reference state means that Wick polynomials are *themselves* not uniquely defined. The ambiguities involved with the definition of these two classes of fields are physically interpreted as *finite renormalizations* or *renormalization counterterms*, upon adopting the natural locally covariant generalization of Epstein-Glaser approach to renormalization.

Exactly as in flat spacetime, each fixed type of (either Wick or time-ordered) polynomial admits a finite-dimensional class of independent counterterms. In curved spacetime, this class is much larger than in Minkowski space, because of the possible dependence of counterterms on *background curvature*. While this class may no longer be finite-dimensional, it is still *finitely generated* or *quasi-finite-dimensional* in a precise sense, because the counterterms may depend only polynomially on the curvature scalars up to a certain dimension. This remarkable result, in the case of Wick polynomials, presented in [11, Thm.5.1] and restated in our Proposition 3.1, is arrived at by imposing severe constraints on Wick polynomials in addition to those of locality and covariance. These requirements are of various kinds. Some arise form heuristic properties of quantum free fields, e.g., Hermiticity and commutation relations. Other requirements concern microlocal features which, loosely speaking, extend to curved spacetime the structure of Fourier transforms of the relevant Green functions on Minkowski space. Another requirement regards the behaviour of Wick polynomials under a rescaling of the metric and the parameters m^2 and ξ of the free theory, which describe the field's mass[1] and its coupling to the curvature. Finally there are the technically delicate requirements of *continuous* and *analytic* dependence on the metric. The two latter requirements play a crucial role in [11] in their proof of the strong restrictions on possible finite renormalization counterterms that was mentioned above.

The main difficulty with defining a suitable notion of the *continuous* dependence of an element of the algebra $\mathcal{W}(M, \mathbf{g})$ on the metric $\mathbf{g}$ (and the other parameters m^2 and ξ) is that, continuously changing the metric $\mathbf{g} \mapsto \mathbf{g}'$, the whole algebra $\mathcal{W}(M, \mathbf{g})$ changes correspondingly and algebras $\mathcal{W}(M, \mathbf{g})$ and $\mathcal{W}(M, \mathbf{g}')$

[1] As in [11], we will always treat m^2 as a real number, which could be either positive, zero, or even negative, as ultraviolet renormalization is not sensitive to the sign of m^2.

associated with different metrics are not canonically isomorphic. Therefore even just stating the condition of continuous dependence on $\mathbf{g}$ turns out to be difficult. Locality can be turned into an advantage in this context [11]. One may restrict attention to metric variations in a spacetime region $O \subset M$ with compact closure. If $\mathbf{g}$ agrees with $\mathbf{g}'$ outside O, essentially exploiting a suitable version of the *time slice axiom*, it is possible to naturally identify an element of $\mathcal{W}(M, \mathbf{g})$ with a corresponding element in $\mathcal{W}(M, \mathbf{g}')$, when both are supported in O. Hence, a local version of the continuity requirement can be imposed by means of this canonical identification.

The requirement of *analytic* dependence is even trickier to state. It is argued in [11] that analytic dependence is necessary because the remaining requirements would not be able to rule out the undesirable infinite family of non-polynomial in curvature counterterms that were considered in [20]. There is an important subtle technical issue arises in stating this analytic dependence condition. The way followed for stating the continuity dependence requirement in a local region O faces here an insurmountable obstruction: analytic metric are *rigid* and if they coincide outside O they must coincide also in O. The ingenious but cumbersome strategy elaborated in [11] a special class of Hadamard states over the considered algebras. Since no local analytic variations of the metric are possible, they consider a joint analytic family $\mathbf{g}^{(s)}$ of the metric on O and a corresponding analytic family of quasifree Hadamard states $\omega^{(s)}$ on $\mathcal{W}(M, \mathbf{g}^{(s)})$. Then they require that the distributions obtained by composing $\omega^{(s)}$ with the local Wick polynomials (or their time ordered products) varies analytically with s in a suitable *analytic and microlocal* sense (see the discussion starting on p.311 in [11]).

Continuous and analytic dependence on the parameters m^2 and ξ is there treated similarly, with both parameters taken to be functions on M, rather than just constants, at intermediate stages of the arguments.

The main result of this work establishes that the technically cumbersome and somewhat unnatural continuous and analytic dependence requirements are in fact *not necessary* to achieve the classification theorem [11, Thm.5.1]. Our classification result, Theorem 3.2, is essentially identical, though it is slightly more general because it allows smooth (rather than just analytic) dependence on the dimensionless curvature coupling ξ. In the proof, we note that, keeping the same algebraic and microlocal requirements, the locality, scaling and covariance requirements are completely sufficient to achieve the desired classification.

The key tool exploited in our proof is a theorem that characterizes (generally non-linear) differential operators in terms of their locality properties. This theorem, known as the *non-linear Peetre theorem*, in its most elementary version (Proposition 2.2; see also Appendix A for a more general statement) states the following: any map D, that associates smooth sections $\psi\colon M \to E$ of a bundle $E \to M$ to smooth sections $D[\psi]\colon M \to F$ of another bundle $F \to M$ in such a way that $D[\psi](x)$ depends only on the germ of ψ at x for any point $x \in M$, is necessarily a differential operator of locally bounded order, smoothly depending on its arguments and their derivatives. The C_k coefficients that characterize renormalization counterterms of Wick polynomials precisely map sections of the bundle of metrics and parameters, m^2 and ξ, to scalar valued distributions on a spacetime M. The microlocal conditions ensure that these distributions are actually smooth functions, while the locality requirement implies that the C_k satisfy the hypotheses of Peetre's theorem and hence must be differential operators. A combination of the scaling and covariance requirements then shows that the differential order of the C_k is globally bounded and that their dependence on the metric, m^2 and the derivatives of all the parameters is polynomial, with coefficients smoothly depending on ξ. Further, covariance also dictates that the derivatives of the metric necessarily group into curvature scalars.

Notably, continuous and analytic dependence requirements are not exploited in establishing the above result. Within the context of our proof, counter terms like $m^k F(R/m^2)$, where R is the Ricci scalar and F is any smooth function with strong decal near 0 and $\pm\infty$, as considered in [20], are excluded because they violate the microlocal requirement: there exists a choice of a spacetime $(M, \mathbf{g})$ and of a scalar field m^2 such that the counterterm is not smooth and hence has non-empty wavefront set and the Wick polynomials modified by adding these counterterms do not satisfy the microlocal requirement.

This paper is organized as follows. Our main theorem and its proof are presented in Section 3. The proof is somewhat lengthy, but straight forward. It relies on some preliminary definitions and results discussed in Section 2. In particular our basic version of Peetre's theorem is stated in Section 2.3 after a quick summary of elementary facts about jet bundles in Section 2.2, where we also introduce some useful coordinate systems. Section 2.4 is devoted to introducing our notion of scaling which is more precise but substantially equivalent to the one employed in [11]. Though, we are careful to identify two different kinds of scalings (physical and coordinate), which were mixed in [11] by the introduction of Riemann normal coordinates. The remainder of Section 2 deals with notions and results, especially on $GL(n)$ representation theory, which are useful for imposing the covariance requirement. After recalling the definition and properties of Wick polynomials, and the more general notion of locally covariant quantum field, with the appropriate categorical language, we state and prove out main result in several steps in Section 3. Section 4 concludes the paper with a discussion of the results and directions for future work. Appendix A illustrates a more general version of Peetre's theorem, which applies to differential operators with parameters.

2 Geometry of scaling and general covariance

In this section we discuss some aspects of the geometry of the higher derivatives (jets) of metric and scalar fields under the action of scaling and diffeomorphism transformations. These properties will be crucial in the characterization of finite renormalizations in locally covariant quantum field theory in Section 3.

2.1 Coordinates on jets

In differential geometry, *jets* [15, 14] are a geometric way of collecting information about higher derivatives of functions (or bundle sections) on manifolds, similar to what the tangent and cotangent bundles do for first derivatives. Jets have an invariant geometric meaning even on manifolds without a preferred metric or connection. Further, a choice of a coordinate chart on a manifold induces a choice of adapted coordinates on the corresponding jet bundle. One advantage of working with jets is that certain calculations are very conveniently performed in such an adapted local coordinate chart, yet also lead to global and geometrically invariant conclusions. Below, we briefly discuss some variations on adapted local coordinate systems on the space of jets of bundle of metrics with some scalar fields.

Consider a smooth map $f\colon \mathbb{R}^m \to \mathbb{R}^n$, such that $f(0) = 0$. The **germ** of f at $0 \in \mathbb{R}^m$ is the equivalence class of smooth maps $f'\colon \mathbb{R}^m \to \mathbb{R}^n$ that agree with f on some neighborhood of $0 \in \mathbb{R}^m$. The r**-jet** of f at $0 \in \mathbb{R}^m$ is the equivalence class of all smooth maps $f'\colon \mathbb{R}^m \to \mathbb{R}^n$ that have the same Taylor expansion at 0 as f to order r, denoted $j^r_0 f$. Obviously, the germ contains more information than a jet of any order. These definitions are clearly local, both on the domain and the target of a smooth map, and are invariant under C^∞-changes of coordinates. Thus, these definitions easily translate to maps between smooth finite-dimensional (smooth) manifolds M, N replacing $0 \in \mathbb{R}^m$ and $0 \in \mathbb{R}^n$, respectively, by generic points $x \in M$, $y \in N$. In particular, with the said M and N, we denote by $J^r(M,N)$ the set of all distinct jets $j^r_x f$ of all smooth maps $f\colon M \to N$ for all $x \in M$. Also, if $E \to N$ is a smooth bundle over N, then we denote by $J^r E$ or, for emphasis, by $J^r(E \to N) \subset J^r(N,E)$ the subset of jets of smooth sections $f\colon N \to E$. Both $J^r(M,N)$ and $J^r(E \to N)$ can be given structures of smooth manifolds. A fiber $(J^r E)_x$ at $x \in N$ is diffeomorphic to $E_x \times \mathbb{R}^{s_r}$, where E_x is the fiber of E and s_r counts the components of all (symmetrized) partial derivatives up to order r. In fact, by projection onto the target of each jet, $J^r E \to E \to N$ is an iterated smooth bundle. Given a section $\psi\colon N \to E$, we can collect the r-jets of ψ over each point of N into a section $j^r\psi\colon N \to J^r E$ called the r**-jet extension** of ψ.

Let (x^a, v^i) be a local adapted coordinate chart on a bundle $F \to M$, where (x^a) serve as coordinates on a domain $U \subseteq M$ and (x^a, v^i) serve as trivializing coordinates on the fibers of the domain $V \subseteq F$ over U. For example, if $T^p_q M \to M$ is the bundle of (p,q)-tensors we can choose coordinates $(x^a, t^{a_1\cdots a_p}_{b_1\cdots b_q})$ on the projection pre-image V of U, such that a section $\tau\colon T^p_q M \to M$ could locally be written as

$$\tau(x) = t^{a_1\cdots a_p}_{b_1\cdots b_q}(\tau(x))\,\mathrm{d}x^{b_1}\cdots \mathrm{d}x^{b_q}\,\frac{\partial}{\partial x^{a_1}}\cdots\frac{\partial}{\partial x^{a_p}}. \tag{1}$$

The local chart (x^a, v^i) then induces an adapted coordinate system (x^a, v^i_A) on the domain $V^r \subseteq J^r E$ that is the projection pre-image of V and is diffeomorphic to $V^r \cong V \times \mathbb{R}^{s_r}$, with s_r as discussed above. Each $A = a_1 \cdots a_l$, standing in for an unordered (equivalently, fully symmetrized) collection of base manifold coordinate indices, is a **multi-index** of size $|A| = l$, with the range $l = 0, 1, \ldots, r$. The defining property of these coordinates is the identity

$$v^i_A(j^r\psi(x)) = \partial_A v^i(\psi(x)) = \frac{\partial}{\partial x^{a_1}}\cdots\frac{\partial}{\partial x^{a_l}} v^i(\psi(x)), \tag{2}$$

for any section $\psi\colon M \to F$. Given such a coordinate system, for brevity, we use the notation $\partial_a = \partial/\partial x^a$ and $\partial^A_i = \partial/\partial v^i_A$ for corresponding coordinate vector fields.

2.2 Coordinates on jets of metric and scalar fields

If M is a n-dimensional smooth manifold, let us now fix the bundle $BM \to M$ given by the bundle product of the bundle $\mathring{S}^2 T_* M$ of (smooth) Lorentzian metric $(0,2)$-tensors over M and the trivial bundle $\mathbb{R} \times M \to M$ of (smooth) scalar fields over M. Let us denote the sections of this bundle by $(\mathbf{g}, \xi)\colon M \to BM$. There are several local coordinate systems on $J^r BM$, of various merits, which we discuss below.

Covariant coordinates. Given a local coordinate chart (x^a) on $U \subseteq M$, we define the corresponding adapted coordinates (x^a, g_{ab}, z) on $V \subseteq BM$, which in turn induce the **covariant coordinates**

$$(x^a, g_{ab,A}, z_A) \quad \text{on } V^r \subseteq J^r BM. \tag{3}$$

Notice that only $n(n+1)/2$ components of g_{ab} take part in the above coordinates, in view of the symmetry of the metric.

Contravariant coordinates. Recall that a Lorentzian metric $\mathbf{g}\colon M \to \mathring{S}^2 T^*M$ is invertible and hence defines a section $\mathbf{g}^{-1}\colon M \to \mathring{S}^2 TM$. The components of the inverse metric can be extracted by functions g^{ab} defined on all of $V \subseteq BM$, such that $g^{ab}(\mathbf{g}^{-1}(x)) = g_{ab}(\mathbf{g}(x))$, which induce the functions g^{ab}_A on V^r that satisfy $g^{ab}_A(j^r \mathbf{g}(x)) = \partial_A g^{ab}(\mathbf{g}(x))$. Then, using the notation $g^{AB} = g^{a_1 b_1} \cdots g^{a_l b_l}$, for $|A| = |B| = l$, we define the following functions

$$g = |\det g_{ab}|, \qquad g^{ab,A} = g^{AB} g^{ab}_B, \qquad z^A = g^{AB} z_A, \tag{4}$$

where, by invertibility of Lorentzian metrics, the function g^{-1} is well defined on all of V^r, since $g = |\det g_{ab}|$ is never zero. These functions make up the alternative set of local **contravariant coordinates**

$$(x^a, g^{ab,A}, z^A) \quad \text{on } V^r \subseteq J^r BM, \tag{5}$$

with the caveat that as the set of functions (g, g^{ab}) is only functionally independent up to the identity $g^{-1} = \left|\det g^{ab}\right|$, for instance, *one of the contravariant coordinates g^{ab} can be replaced by g.* These coordinates have convenient scaling properties that will be exploited in Section 2.4.

Rescaled contravariant coordinates. Another coordinate system that we introduce on $V^r \subseteq J^r BM$, the **rescaled contravariant coordinates**, is a suitable rescaling of the previous one. Namely, we introduce various factors of g^α in the latter coordinates (n being the dimension of M):

$$(x^a, g, g^{-\frac{1}{n}} g_{ab}, g^{\frac{1}{n} + \frac{1}{n}|A|} g^{ab,A}, g^{\frac{s}{2n} + \frac{1}{n}|A|} z^A), \tag{6}$$

where one of the $n(n+1)/2$ functions $g^{-\frac{1}{n}} g_{ab}$ is omitted and replaced by g. This is because the functions $g^{-\frac{1}{n}} g_{ab}$ are not functionally independent because of the relation $|\det g^{-\frac{1}{n}} g^{ab}| = 1$.

Curvature coordinates. Recall also that, given a Lorentzian metric $\mathbf{g}$, we can always define the corresponding covariant derivative, or Levi-Civita connection, ∇ and the Riemann tensor $\mathbf{R}$. Using well known formulas, we can define functions Γ^a_{bc} and $\bar{R}_{abcd}$ on $V^r \subseteq J^r BM$ that correspond to the coordinate components of the Christoffel symbols and the fully covariant Riemann tensor. Define also the fully contravariant tensor $\mathbf{S}$ with components

$$\bar{S}^{abcd} = g^{aa'} g^{bb'} \bar{R}_{a'}{}^{(c}{}_{b'}{}^{d)} = g^{ab,cd} - g^{b(c,d)a} - g^{a(d,c)b} + g^{cd,ab} + \text{l.o.t}, \tag{7}$$

where l.o.t stands for terms that involve only coordinates of lower derivative order. Finally, let $\Gamma^a_{bc,A}$ denote the components of the coordinate ∂_A derivatives of Γ^a_{bc}, let $\bar{S}^{abcd,A}$ denote the components of the symmetrized contravariant $\nabla^A = \nabla^{(a_1} \cdots \nabla^{a_l)}$ derivatives of $\mathbf{S}$, and let $\bar{z}^A$ the components of the symmetrized contravariant ∇^A derivatives of the scalar field ξ. It is well-known [13, 1] that

$$(x^a, g_{ab}, \Gamma^a_{(bc,A)}, \bar{S}^{ab(cd,A)}, \bar{z}^A) \tag{8}$$

also defines a coordinate system on $V^r \subseteq J^r BM$, which we shall call **curvature coordinates**. Note that the barred coordinate functions correspond to components of fully contravariant tensors. These coordinate have convenient transformation properties under diffeomorphisms that will be exploited in Section 2.5.

Rescaled curvature coordinates. The final coordinate system that we introduce on $V^r \subseteq J^r BM$, the **rescaled curvature coordinates**, merges some of the properties of the systems $(x^a, g^{-\frac{1}{n}} g_{ab}, g^{\frac{1}{n}+\frac{1}{n}|A|} g^{ab,A}, g^{\frac{s}{2n}+\frac{1}{n}|A|} z^A)$ and $(x^a, g_{ab}, \Gamma^a_{(bc,A)}, \bar{S}^{ab(cd,A)}, \bar{z}^A)$. Namely, we again introduce various factors of g in the curvature coordinates:

$$(x^a, g, g^{-\frac{1}{n}} g_{ab}, \Gamma^a_{(bc,A)}, g^{\frac{3}{n}+\frac{1}{n}|A|} \bar{S}^{ab(cd,A)}, g^{\frac{s}{2n}+\frac{1}{n}|A|} \bar{z}^A), \tag{9}$$

where, again, one of the $n(n+1)/2$ functions $g^{-\frac{1}{n}} g_{ab}$ is omitted and replaced by g.

2.3 Locality and the non-linear Peetre theorem

It is well known that *linear* differential operators have the property that they are *support non-increasing*. The powerful, original result of Peetre [16, 17] shows that this property is sufficient to characterize them in the context of C^∞ differential geometry. A similar characterization holds even for *non-linear* differential operators [19, 14], a version of which we present below.

Before proceeding, we need a robust geometric notion of what a *differential operator* is. Often, differential operators are defined by their expressions in coordinate charts. Any such definition is necessarily coordinate dependent and must be checked to agree on chart overlaps. On the other hand, we can give a *coordinate independent and global* definition of differential operators using jets and the r-jet extension map j^r defined earlier in Section 2.2.

Given a smooth bundle $E \to N$, recall that the r-jet extension acts as a map $j^r : \Gamma(E \to N) \to \Gamma(J^r E \to N)$, where as usual $\Gamma(G \to L)$ denotes the space of *smooth sections* of the bundle $G \to L$. For our purposes, the map j^r will serve as a universal differential operator of order r in the following sense.

Definition 2.1. Let $E \to N$ and $F \to M$ be smooth bundles, and consider a map $D\colon \Gamma(E) \to \Gamma(F)$.

(a) D is a **differential operator of globally bounded order** if there exists an integer $r \geq 0$, the **order**, and a smooth function $d\colon J^r(E \to N) \to F$, considered as a bundle map (i.e., fiber preserving), such that for any section $\psi \in \Gamma(E)$ we have an associated section of the form $D[\psi] = d \circ j^r \psi \in \Gamma(F)$.

(b) D is a **differential operator of locally bounded order** if it satisfies a similar condition locally. Namely, for any point of $y \in N$ and section $\phi \in \Gamma(E)$, there exists a neighborhood $U \subseteq N$ of y with compact closure, together with an integer $r \geq 0$, an open neighborhood $V^r \subseteq J^r(E \to N)$ of $j^r\phi(U)$ projecting onto U, and a smooth function $d\colon V^r \to F$ that respects the projections $V^r \to U$ and $F \to M$, such that $D[\psi](x) = d \circ j^r\psi(x)$ for any $x \in U$ and any $\psi \in \Gamma(E)$ with $j^r\psi(U) \subset V^r$.

If $E \to M$ and $F \to M$ are vector bundles over the same base manifold M and $D\colon \Gamma(E) \to \Gamma(F)$ is a *linear* map such that $\phi(x) = D[\psi](x)$ depends only on the germ of ψ at $x \in M$ then it is clear that D will be support non-increasing. Elementary reasoning shows that a linear, support non-increasing map will also only depend on germs. So, another way to rephrase the Peetre theorem for linear differential operators is as follows, where the dependence on the germ replaces the support non-increasing property.

Proposition 2.1 (Linear Peetre's Theorem [16, 17])**.** *Let $E \to M$ and $F \to M$ be vector bundles and $D\colon \Gamma(E) \to \Gamma(F)$ a linear map such that $\phi(x) = D[\psi](x)$ depends only on the germ of ψ at $x \in M$. Then D is a linear differential operator of locally bounded order (with smooth coefficients in view of the above definition).*

In other words, despite the fact that germs potentially contain much more information that jets, such a linear map that depends only on germs in fact sees only jets.

Phrased as above, in terms of germs, the hypotheses of Peetre's theorem are immediately adaptable to the case when the map D is non-linear and acts on sections of (non-vector) smooth bundles.

Proposition 2.2 (Non-linear Peetre's Theorem [14, §19])**.** *Let $E \to M$ and $F \to M$ be smooth bundles and $D\colon \Gamma(E) \to \Gamma(F)$ map such that $\phi(x) = D[\psi](x)$ depends only on the germ of ψ at $x \in M$. Then D is a (non-linear) differential operator of locally bounded order.*

This proposition will be sufficient for our purposes. However, in the standard literature [19, 14], this result is stated in much greater generality. In fact, that level of generality can obscure the meaning and significance of the theorem. In Appendix A, we briefly introduce the language needed to state a more general version, Proposition A.1. The above simpler version becomes a special case of Proposition A.1 once it is trivially checked that D is id-local, where $\mathrm{id}\colon M \cong M$ is the identity map. The more general result given in Appendix A serves two purposes. The first is that its introduces the language in which the non-linear Peetre theorem and its proof appear in the standard literature [14, §19]. Second, it allows the treatment of differential operators with parameters. For instance, later in Section 3, we treat the mass m^2 of a scalar field and its coupling to curvature ξ as space-time dependent background fields. If they were treated as necessarily spacetime-constant parameters, we would need to substitute Proposition A.1 for the simpler Proposition 2.2 in the proof of our main Theorem 3.2.

2.4 Physical scaling

Referring to the already introduced bundle $BM \to M$, sections $(\mathbf{g}, \xi) \in \Gamma(BM)$ consist of a smooth Lorentzian metric $\mathbf{g}$ and a smooth scalar field ξ on M. We consider the following scaling transformation $(\mathbf{g}, \xi) \mapsto (\lambda^{-2}\mathbf{g}, \lambda^s\xi)$ on sections. We call this transformation a **physical scaling**, in contrast to a different kind of scaling to be introduced in Section 2.5. We will need the following rather general *recursive* definition, where $\mathbb{R}^+ := (0, +\infty)$,

Definition 2.2. Consider a linear representation of the multiplicative group $\mathbb{R}^+$ on a vector space W, written as $W \ni F \mapsto F_\lambda \in W$, for every $\lambda \in \mathbb{R}^+$.

(a) An element $F \in W$ is said to have **homogeneous degree** $k \in \mathbb{R}$ if

$$F_\lambda = \lambda^k F \quad \text{for all } \lambda \in \mathbb{R}^+ . \tag{10}$$

(b) An element $F \in W$ is said to have **almost homogeneous degree** $k \in \mathbb{R}$ **and order** $l \in \mathbb{N}$ if $l \geq 0$ is an integer such that (the sum over j is omitted if $l = 0$)

$$F_\lambda = \lambda^k F + \lambda^k \sum_{j=1}^{l} (\log^j \lambda) G_j, \quad \text{for all } \lambda \in \mathbb{R}^+, \tag{11}$$

and for some $G_j \in W$ depending on F, which have respectively almost homogeneous degree k and order $l - j$.

The definition is recursive, with higher orders defined in terms of lower ones. Clearly, an element that is almost homogeneous of order $l = 0$ is simply homogeneous.

Remark 2.1. Besides *almost homogeneous*, other common names found in the literature include *poly-homogeneous*, *associated homogeneous* and even *quasi associated homogeneous*. We are mostly interested in the case when W is some function space and the action or $\mathbb{R}^+$ is induced from an action on the domain of the functions. Reference [18] reviews several definitions leading to this class of functions and lists relevant earlier works. In the context of distribution theory, the terminology of *associated homogeneous* is prevalent and goes back to the seminal references [8, §1.4] and [9, Ch.I §4].

The **physical scaling transformation** on the sections $\Gamma(BM)$ can be implemented by post-composing a section with a bundle map $BM \to BM$:

$$BM \ni (p, \mathbf{g}(p), z(p)) \mapsto (p, \lambda^{-2}\mathbf{g}(p), \lambda^s z(p)) \in BM , \tag{12}$$

where the real $\lambda \in \mathbb{R}^+$ defines the scaling transformation. This representation of the multiplicative group $\mathbb{R}^+$ is *globally* defined, however this global action can be written in adapted local coordinates, as discussed in Section 2.2, and looks like

$$x^a \mapsto x^a, \quad g_{ab} \mapsto \lambda^{-2} g_{ab}, \quad z \mapsto \lambda^s z. \tag{13}$$

This global transformation lifts to a global transformation of the jet bundle $J^r BM$. In the corresponding induced local coordinates, the lifted action reads

$$g_{ab,A} \mapsto \lambda^{-2} g_{ab,A}, \quad z_A \mapsto \lambda^s z_A. \tag{14}$$

We are interested in applying Definition 2.2 to $W = C^\infty(J^r BM)$ and the $\mathbb{R}^+$ action induced by the lift of physical scalings to $J^r BM$. Moreover, we will need to consider also smaller domains $V^r \subseteq J^r BM$ for these functions, with V^r themselves not invariant under physical scalings. Thus, it is more convenient to refer to the infinitesimal version of these transformations, which are effected by the following vector field

$$e = -2 g_{ab,A} \partial^{ab,A} + s z_A \partial_z^A, \tag{15}$$

in the sense that the induced action on scalar functions on J^rBM satisfies

$$\frac{d}{d\lambda}\bigg|_{\lambda=0} F_\lambda = \mathcal{L}_e F. \tag{16}$$

(In the rest of the paper if X is a vector field on J^rBM, $\mathcal{L}_X$ denotes the standard Lie derivative so that, in particular $\mathcal{L}_X(F) := X(F)$ if $F\colon V^r \subseteq J^rBM \to \mathbb{R}$ is a smooth function.) Notice that, as the physical scaling transformation is globally defined, e turns out to be globally defined on J^rBM and (15) is just its expression in local coordinates. We have a first elementary result stated within the following lemma. We will essentially show later that the converse implication holds as well.

Lemma 2.3. *A smooth function $F\colon J^rBM \to \mathbb{R}$ that has almost homogeneous degree k, according to Definition 2.2, when the action $F \to F_\lambda$ is the one induced by physical scaling transformations, satisfies the following local infinitesimal version*

$$(\mathcal{L}_e - k)^{l+1} F = 0\,. \tag{17}$$

Proof. It is sufficient to make use of equation (16) and recall the obvious identity $(d/d\lambda - k)^{l+1}\lambda^k \log^l \lambda = 0$. □

This lemma permits us to state the following

Definition 2.3. A smooth function $F\colon V^r \subseteq J^rBM \to \mathbb{R}$, where V^r is an open subset which may coincide with all of J^rBM, is said to have **almost homogeneous degree** $k \in \mathbb{R}$ **and order** $l \in \mathbb{N}$ (with $l \geq 0$) **under physical scalings** if it satisfies the identity

$$(\mathcal{L}_e - k)^{l+1} F = 0. \tag{18}$$

If $l = 0$, F is said to have **homogeneous degree** $k \in \mathbb{R}$.

To investigate the local structure of F above we initially use an open subset V^r equipped with the contravariant coordinates $(x^a, g^{ab,A}, z^A)$ introduced in Section 2.2. In these coordinates, finite and infinitesimal physical scalings take the form

$$x^a \mapsto x^a, \quad g \mapsto \lambda^{-2n} g, \quad g^{ab,A} \mapsto \lambda^{2+2|A|} g^{ab,A}, \quad z^A \mapsto \lambda^{s+2|A|} z^A, \tag{19}$$

$$e = (2 + 2|A|) g^{ab,A} \partial_{ab,A} + (s + 2|A|) z^A \partial^z_A\,, \tag{20}$$

where we have also described the action of rescaling on g which, as already remarked, can be used as an alternative coordinate in place of one of the g^{ab}. As e does not vanish anywhere, J^rBM and hence the domain V^r are foliated by integral curves of the vector field e. Moreover, the identity $\mathcal{L}_e g^{-\frac{1}{2n}} = g^{-\frac{1}{2n}}$ means that g restricts to a global coordinate on each orbit of e. Thus, the level sets of g constitute another foliation of J^rBM and V^r, transverse to the integral curves of e. These observations suggest to study the structure of (almost) homogeneous functions of degree k in the *rescaled contravariant coordinates*

$$(x^a, g, g^{-\frac{1}{n}} g_{ab}, g^{\frac{1}{n}+\frac{1}{n}|A|} g^{ab,A}, g^{\frac{s}{2n}+\frac{1}{n}|A|} z^A), \tag{21}$$

that were introduced in Section 2.2. Note that *each of these functions but g is invariant under physical scalings*. We have the following result.

Lemma 2.4. *Suppose that $V^r \subseteq J^r BM$ is an open set equipped with either coordinates $(x^a, g^{ab,A}, z^A)$ or some other coordinate system introduced in Section 2.2, and $F\colon V^r \to \mathbb{R}$ is a smooth function that has almost homogeneous of degree k and order l with respect to physical scalings, as in Definition 2.3. Then there exist homogeneous of degree k functions $H_j\colon V^r \to \mathbb{R}$, for $j = 0, 1, \ldots, l$, such that*

$$F = g^{-\frac{k}{2n}} \sum_{j=0}^{l} \log^j(g^{-\frac{1}{2n}}) H_j. \tag{22}$$

In particular, using rescaled contravariant coordinates, each H_j can be taken independent of g and written in the form

$$H_j = H_j(x^a, g^{-\frac{1}{n}} g_{ab}, g^{\frac{1}{n}+\frac{1}{n}|A|} g^{ab,A}, g^{\frac{s}{2n}+\frac{1}{n}|A|} z^A). \tag{23}$$

Proof. In the simplest $l = 0$ case, we can define $H = g^{\frac{k}{2n}} F$ and show that $\mathcal{L}_e H = 0$ because $\mathcal{L}_e g^{-\frac{1}{2n}} = g^{-\frac{1}{2n}}$. This means that, in rescaled contravariant coordinates, H is independent of g and hence (23) holds, with H in place of H_j. Next, the general $l \geq 1$ case can be treated as follows. Let $G := g^{\frac{k}{2n}} F$, which implies that $\mathcal{L}_e^l G = 0$. Now, note the identity $\mathcal{L}_e^j \log^j(g^{-\frac{1}{2n}}) = j!$. So, if $H_l := \frac{1}{l!} \mathcal{L}_e^l G$ and $G_{l-1} := G - \log^l(g^{-\frac{1}{2n}}) H_l$, then $\mathcal{L}_e H_l = 0$ and $\mathcal{L}_e^l G_{l-1} = 0$. In other words, starting with $G_l = G$, we can recursively define $H_j := \frac{1}{j!} \log^j(g^{-\frac{1}{2n}}) \mathcal{L}_e^j G_j$ and $G_{j-1} := G_j - \log^j(g^{-\frac{1}{2n}}) H_j$, finding $\mathcal{L}_e H_j = 0$ at each step. The procedure stops for $j = 0$ when it gives $G_0 = H_0$, so that $G_{j<0} = H_{j<0} = 0$, proving (22). □

We will also need the following basic result regarding *products* of vectors with almost homogeneous degree as in Definition 2.2. Due to the generality of Definition 2.2 we must clarify the meaning of *product*. If W and W' are two vector spaces, by a **product** between them, we mean any fixed bilinear map $W \times W' \to V$, where V is another vector space. If $F \in W$ and $F' \in W'$ the corresponding element in V, their **product**, will be simply denoted by $FF' \in V$.

Lemma 2.5. *Referring to Definition 2.2, consider a pair of vector spaces W, W' endowed with corresponding representations of $\mathbb{R}^+$. Concerning (b) below, assume also that there is a product $W \times W' \to V$ such that (i) V admits a representation of $\mathbb{R}^+$ and (ii) the map $W \times W' \to V$ is equivariant: $F_\lambda F'_\lambda = (FF')_\lambda$ for $F \in W$, $F' \in W'$ and $\lambda \in \mathbb{R}^+$.*

- ***(a)*** *A linear combination of two elements $F, F' \in W$ of almost homogeneous degree k and order l is of almost homogeneous degree k and order l.*
- ***(b)*** *A product of an element $F \in W$, of almost homogeneous degree k and order l, and an element $F' \in W'$, of almost homogeneous degree k' and order l', has almost homogeneous degree $k + k'$ and order $l + l'$.*

Proof. Part (a) is trivial, because the defining identity 11 is linear.

We will prove part (b) by double induction on the pair of orders (l, l').

Consider the identity

$$(FF')_\lambda = F_\lambda F'_\lambda = \lambda^{k+k'} FF' \\ + \lambda^{k+k'} \sum_{j=1}^{l} (\log^j \lambda) G_j F' + \lambda^{k+k'} \sum_{j'=1}^{l'} (\log^{j'} \lambda) F G'_{j'} \\ + \lambda^{k+k'} \sum_{j=1}^{l} \sum_{j'=1}^{l'} (\log^{j+j'} \lambda) G_j G'_{j'}. \quad (24)$$

From this formula, it is clear that, to show that FF' has almost homogeneous degree $k+k'$ and order $l+l'$, it is sufficient to establish that the coefficients of the logarithmic terms, $G_j F'$, $FG'_{j'}$ and $G_j G'_{j'}$, either do not appear or are themselves almost homogeneous of the right degree and order. Thus, to establish the case (l,l'), it is sufficient to have all of the (j,l'), (l,j') and (j,j') cases, with $j < l$ and $j' < l'$, already established. We shall refer to this last remark as the *primary inductive step*.

The case $(l,l') = (0,0)$ follows immediately from Equation (24), since no logarithmic terms appear. Next, we establish the following *secondary inductive step*. Assuming that, given some $m \geq 0$, all cases (l,l') with $l,l' \leq m$ hold, then actually all cases (l,l') with $l,l' \leq m+1$ hold as well. To see that, note that the case $(m+1,0)$ holds, because in (24) we need only consider the terms $G_j F'$, which correspond to the inductively covered cases $(m+1-j,0)$ with $j \geq 1$. Then, using the primary inductive step, all the cases $(m+1,l')$ with $1 \leq l' \leq m$ follow as well. The cases $(l,m+1)$ with $0 \leq l \leq m$, are completely analogous. Finally, one more appeal to the primary inductive step establishes the case $(m+1,m+1)$.

Iterating the secondary inductive step completes the proof of part (b). □

2.5 Diffeomorphisms and coordinate scalings

Because the sections $(\mathbf{g},\xi) \in \Gamma(BM)$ are tensor fields, there is a well defined action of the group $\mathrm{Diff}(M)$ of diffeomorphisms $\chi\colon M \to M$ on them by pullback $(\mathbf{g},\xi) \mapsto (\chi^*\mathbf{g}, \chi^*\xi)$. This action of course can be implemented at the level of the bundle itself, $\chi^*\colon BM \to BM$ and of course lifted to the jet bundle $j^r\chi^*\colon J^rBM \to J^rBM$. We are interested in the structure of functions $F\colon J^rBM \to \mathbb{R}$ that are invariant under the action of $\mathrm{Diff}(M)$. We could also consider invariance only under the subgroup $\mathrm{Diff}^+(M)$ of orientation preserving diffeomorphisms in an essentially analogous way. For this purpose, it is convenient to make use of the local adapted *curvature coordinates* $(x^a, g_{ab}, \Gamma^a_{(bc,A)}, \bar{S}^{ab(cd,A)}, \bar{z}^A)$ on a domain $V^r \subseteq J^rBM$ defined in Section 2.2.

The domain V^r itself may not be invariant under $\mathrm{Diff}(M)$, because our coordinates are adapted to a single coordinate chart (x^a) on $U \subseteq M$. On the other hand, having already chosen our coordinate system, we can phrase the requirement that $F\colon V^r \to \mathbb{R}$ is the restriction of a $\mathrm{Diff}(M)$-invariant function (necessarily defined on a possibly larger $\mathrm{Diff}(M)$-invariant domain) to V^r in the following way: (a) $\frac{\partial}{\partial x^a} F = 0$, where the vector fields $\frac{\partial}{\partial x^a}$ are the infinitesimal generators of diffeomorphisms that restrict to coordinate translations on U, and (b) the restriction $F_x\colon V^r_x \subseteq J^r_x BM \to \mathbb{R}$ of F to the fiber of J^rBM over any one point $x \in M$ is invariant under the action of the subgroup $\mathrm{Diff}(M,x) \subset$

Diff(M) that fixes x. Clearly we can take V_x^r to be invariant under Diff(M,x). An immediate simplification based on requirement (a) is that our function is expressible as $F = F_x(g_{ab}, \Gamma^a_{(bc,A)}, \bar{S}^{ab(cd,A)}, \bar{z}^A)$, that is, it is independent of the base coordinates (x^a). Next, we examine the consequences of requirement (b).

The action of Diff(M,x) on r-jets is not faithful. In fact, it has a large kernel, so that the action on $J_x^r BM$ factors through the homomorphic projection Diff$(M,x) \to G_n^r$, where G_n^r is a finite-dimensional Lie group known as the **r-jet group** [14, §13]. Thus, we need only consider the invariance of F_x under G_n^r. The r-jet groups come with natural projections $G_n^r \to G_n^{r-1}$, corresponding to the equivariant projection $J_x^r BM \to J_x^{r-1} BM$, and it is easily seen that $G_n^1 \cong GL(n)$. Analogously, for orientation preserving diffeomorphisms, we denote the corresponding projections as Diff$^+(M) \to G_n^{+r} \to GL^+(n)$.

The curvature coordinates $(g_{ab}, \Gamma^a_{(bc,A)}, \bar{S}^{ab(cd,A)}, \bar{z}^A)$ specifically for their transformation properties under G_n^r. Note that, without loss of generality but after a possible small restriction of V_x^r, we can factor $V_x^r \cong \mathbb{R}^\gamma \times W^r$, where the projection onto the $\mathbb{R}^\gamma$ factor is effected by the $(\Gamma^a_{(bc,A)})$ coordinates and the projection onto the W^r factor is effected by the remaining coordinates. This factorization respects the action of G_n^r in the sense that the projection $V_x^r \to W^r$ induces a well-defined action of G_n^r and W^r. The action on the action on W^r actually factors through the projection $G_n^r \to G_n^1 \cong GL(n)$, since it is coordinatized by components of tensors. Moreover, for any $w \in W^r$, the isotropy subgroup of w in G_n^r acts transitively on the fiber $\mathbb{R}^\gamma$ over w. In the orientation preserving case, the same is true of the corresponding actions of G_n^{+r} and $GL^+(n)$. The fact that G_n^r (and also G_n^{+r}) acts transitively on the $\mathbb{R}^\gamma$ fibers that are coordinatized by the derivatives of the Christoffel symbols $(\Gamma^a_{(bc,A)})$ means that an invariant function F_x cannot depend on these coordinates, which is a well-known result that is sometimes known as the *Thomas replacement theorem* [13, 1]. Let us rephrase it slightly below.

The above factorization $V^r \cong \mathbb{R}^\gamma \times W^r$ is also compatible with the rescaled curvature coordinates

$$(x^a, g, g^{-\frac{1}{n}} g_{ab}, \Gamma^a_{(bc,A)}, g^{\frac{3}{n}+\frac{1}{n}|A|} \bar{S}^{ab(cd,A)}, g^{\frac{s}{2n}+\frac{1}{n}|A|} \bar{z}^A), \tag{25}$$

that were introduced in Section 2.2. Recall that in our notation the functions $(g^{-\frac{1}{n}} g_{ab})$ are functionally independent only up to the identity $|\det(g^{-\frac{1}{n}} g_{ab})| = 1$. The main distinction is that these coordinates, other than $(x^a, \Gamma^a_{(bc,A)})$, are no longer components of tensors, but rather of tensor densities, which also transform under $GL(n)$ (cf. Section 2.6). Using these coordinates, together with the preceding discussion, we can simplify a Diff(M)-invariant F as follows:

Proposition 2.6 (Thomas replacement theorem)**.** *Let $F\colon V_x'^r \subseteq J^r BM \to \mathbb{R}$ be a* Diff(M)*-invariant function defined on a* Diff(M)*-invariant domain. In the coordinate system* (25) *defined on the domain $V^r \subseteq V'^r$, the restriction of F to V^r must be expressible as*

$$F = G(g, g^{-\frac{1}{n}} g_{ab}, g^{\frac{3}{n}+\frac{1}{n}|A|} \bar{S}^{ab(cd,A)}, g^{\frac{s}{2n}+\frac{1}{n}|A|} \bar{z}^A), \tag{26}$$

where the function G is invariant under the action of $GL(n)$ on its arguments.

At this point, we have reduced the invariance of F under Diff(M) to the invariance of the function G, from Proposition 2.6, under $GL(n)$ (obtained as the projection $\mathrm{Diff}(M, x) \to GL(n)$), which follows from the preceding discussion. Analogous statements hold for $\mathrm{Diff}^+(M)$, $\mathrm{Diff}^+(M, x)$ and $GL^+(n)$. We now single out a specific subgroup of $GL^+(n)$ (and hence also of $GL(n)$) that we shall call the group of **coordinate scalings**. It consists of matrices of the form $\mu I_n \in GL(n)$, where μ is a positive real number and I_n is the $n \times n$ identity matrix. The name refers to the fact that μI_n is the image of a diffeomorphism that restricts to a uniform scaling of the coordinates (x^a) centered at $x \in U \subseteq M$, with of course many other possible pre-images, under the projection $\mathrm{Diff}^+(M) \to GL^+(n)$. These transformations should be contrasted with the distinct group of *physical scalings* introduced in Section 2.4.

Coordinate scalings act on the components of tensor densities appearing in the coordinate system (25) as follows:

$$g \mapsto \mu^{2n} g, \qquad g^{\frac{3}{2}+\frac{1}{n}|A|}\bar{S}^{ab(cd,A)} \mapsto \mu^{2+|A|} g^{\frac{3}{2}+\frac{1}{n}|A|}\bar{S}^{ab(cd,A)}, \tag{27}$$

$$g^{-\frac{1}{n}} g_{ab} \mapsto g^{-\frac{1}{n}} g_{ab}, \qquad g^{\frac{s}{2n}+\frac{1}{n}|A|}\bar{z}^{A} \mapsto \mu^{s+|A|} g^{\frac{s}{2n}+\frac{1}{n}|A|}\bar{z}^{A}. \tag{28}$$

We stress a fundamental difference between *coordinate scalings* and the previously introduced *physical scalings*: coordinate scalings are induced from the action of the diffeomorphism group, while the physical ones are not.

2.6 Equivariant and Isotropic tensors

In this section, we present some basic facts about *equivariant* maps between spaces that carry certain representations of $GL(n)$.

In particular, consider the space B_n of bilinear forms on $\mathbb{R}^n$, and the natural linear action of $GL(n)$ thereon. The subset $L_n \subset B_n$ of non-degenerate bilinear forms of Lorentzian signature $(-+\cdots+)$ is invariant and hence inherits an action of $GL(n)$ itself. If $\eta \in L_n$ is the **canonical Lorentzian form**, defined by the matrix $\mathrm{diag}(-1, 1, \ldots, 1)$ referring to the canonical basis of $\mathbb{R}^n$, the subgroup $O(1, n-1) \subset GL(n)$ is defined as the *isotropy group* of η. We could also restrict the action on L_n to the subgroup $GL^+(n) \subset GL(n)$ of *orientation preserving* transformations. With this choice, the isotropy group of η turns out to be $SO(1, n-1) = O(1, n-1) \cap GL^+(n)$.

Remark 2.2. L_n consists of a single orbit and is in fact isomorphic to the homogeneous space $GL(n)/O(1, n-1)$. Similarly, L_n is also isomorphic to the homogeneous space $GL^+(n)/SO(1, n-1)$. The fact that the action of $GL(n)$ (resp. $GL^+(n)$) is transitive on L_n implies, as a general well-known fact, that the isotropy group of any $g \in L_n$ is isomorphic to $O(1, n-1)$ (resp. $SO(1, n-1)$).

Definition 2.4. Let M_n^p be the space of p-multilinear forms on $\mathbb{R}^n$ and consider the natural linear action of $GL(n)$ thereon. Let T be a finite-dimensional real vector space carrying a representation of $GL(n)$.

(a) T is a **(covariant) tensor representation** if it is the restriction of the action of $GL(n)$ on M_n^p with respect to some linear embedding $T \hookrightarrow M_n^p$ as an invariant subspace. We call p the tensor **rank** of T.

(b) T a **(covariant) tensor density representation** if T is as in (a) but the action of $GL(n) \ni u \mapsto \rho(u)$ on T is given by a tensor representation up to a multiplication by $|\det u|^s$, where s is the tensor **weight** of T.

Of course, we obtain similar definitions by substituting $GL^+(n)$ for $GL(n)$, and also $O(1,n-1)$ or $SO(1,n-1)$, when a particular Lorentzian bilinear form g is fixed. Of course, in the case of $O(1,n-1)$ and $SO(1,n-1)$, there is no distinction between *tensor* and *tensor density* representations.

Finally, it is useful to consider the one point space $* \cong \mathbb{R}^0$ with the trivial action of $GL(n)$ or any of its subgroups thereon.

Definition 2.5. Let X and Y be spaces carrying respective actions $\rho^{(X)}$ and $\rho^{(Y)}$ of the group G. A map $f\colon X \to Y$ is said to be **equivariant** if commutes with the action of G:

$$f \circ \rho_u^{(X)} = \rho_u^{(Y)} \circ f \quad \text{for every } u \in G\,. \tag{29}$$

Consider the special case where $X := *$, $Y := T$ as in (a) in definition 2.4, and $G := O(1,n-1)$. The image of an equivariant map $* \to T$ is called an $O(1,n-1)$**-isotropic tensor**. The space of $O(1,n-1)$-isotropic tensors in T will be denoted by $\mathcal{I}_T$.

An $SO(1,n-1)$**-isotropic tensor** is defined similarly, replacing $O(1,n-1)$ by $SO(1,n-1)$ everywhere. The space of $SO(1,n-1)$-isotropic tensors in T will be denoted by $\tilde{\mathcal{I}}_T$.

Remark 2.3.

(1) The embedding $T \hookrightarrow M_n^p$ is an evident example of equivariant map for $GL(n)$ (and every subgroup) by definition.

(2) As $f\colon * \to T$ is completely defined by its image $f(*) = t \in T$ the definition states that a tensor $t \in T$ is isotropic if it is *invariant* under the relevant action of $O(1,n-1)$ (or $SO(1,n-1)$) on T.

(3) The space of isotropic tensors for different Lorentzian bilinear forms are clearly isomorphic.

It is well known that the subspaces of isotropic tensors $\mathcal{I}_T \subset T$ and $\tilde{\mathcal{I}}_T \subset T$ can be fully characterized as in the proposition below. In the following, $\epsilon \in M_n^n$ denotes the **canonical Levi-Civita tensor**, that is, the fully anti-symmetric form uniquely fixed by the value of its component $\epsilon_{1\cdots n} = 1$, with respect to the canonical basis of $\mathbb{R}^n$. Also, $\mathcal{I}_n^p \subset M_n^p$ will denote the subspace spanned by all possible tensor products of the canonical Lorentzian form $\eta \in L_n$ that create a p-multilinear form. More precisely, $\mathcal{I}_n^p$ is spanned by elements of the form

$$(\eta_\sigma)_{i_1 i_2 \cdots i_{p-1} i_p} = \eta_{\sigma(i_1)\sigma(i_2)} \cdots \eta_{\sigma(i_{p-1})\sigma(i_p)}, \tag{30}$$

where $\sigma \in S_p$ is any permutation. Similarly, $\tilde{\mathcal{I}}_n^p \subset M_n^p$ denotes the subspace spanned by all possible tensor products of η and ϵ that create a p-multilinear form.

Proposition 2.7. *Given a real vector space T carrying a tensor representation of $GL(n)$ and identifying T with its image with respect to the embedding $\alpha : T \hookrightarrow M_n^p$, the following facts hold.*
(a) *The subspace $\mathcal{I}_T \subset T$ is given by $\mathcal{I}_T \cong \alpha(T) \cap \mathcal{I}_n^p$.*
(b) *The subspace $\tilde{\mathcal{I}}_T \subset T$ is given by $\tilde{\mathcal{I}}_T \cong \alpha(T) \cap \tilde{\mathcal{I}}_n^p$.*

An elementary proof of such a characterization of $O(n)$- and $SO(n)$-isotropic tensors can be found in [2], which generalizes straightforwardly to $O(1, n-1)$ and $SO(1, n-1)$. More generally, this kind of result is sometimes known as *first fundamental theorem* of *invariant theory* [21, 10] for the corresponding group.

Definition 2.6. Given a real vector space T with a tensor density representation of $GL(n)$ (resp. $GL^+(n)$) and the natural representation on L_n, we will refer to an equivariant map $t : L_n \to T$ as a **$GL(n)$-equivariant tensor density**, and similarly for **$GL^+(n)$-equivariant tensor densities**.

The space of $GL(n)$-equivariant tensor densities will be denoted by $\mathcal{E}_T$ and the space of $GL^+(n)$-equivariant tensor densities will be denoted by $\tilde{\mathcal{E}}_T$.

Remark 2.4. Even if the functions belonging to $\mathcal{E}_T$ and $\tilde{\mathcal{E}}_T$ are not required to be linear, these spaces enjoy a natural structure of *real vector space*, just in view of the fact that the equivariant tensor densities are maps with values in the real vector space T.

The following lemma characterizes the space of equivariant tensor densities (in the sense of equivariant maps) in terms of isotropic tensors (in the sense of the subspaces $\mathcal{I}_T \subseteq T$ (resp. $\tilde{\mathcal{I}}_T \subseteq T$) defined earlier).

Lemma 2.8. *Let T be a finite-dimensional real vector space carrying a tensor density representation of $GL(n)$, resp. $GL^+(n)$, and assume that L_n is equipped with the natural representation.*

(a) *The space of $GL(n)$-equivariant, resp. $GL^+(n)$-equivariant, tensor densities is isomorphic the subspace of $O(1, n-1)$-isotropic tensors, resp. $SO(1, n-1)$-isotropic tensors, in T. More precisely, the isomorphism is defined by*

$$\mathcal{E}_T \ni t \mapsto t(\eta) \in \mathcal{I}_T \quad (\textit{resp.} \quad \tilde{\mathcal{E}}_T \ni t \mapsto t(\eta) \in \tilde{\mathcal{I}}_T). \tag{31}$$

(b) *For a given $t \in \mathcal{E}_T$, we have*

$$t(g) = |\det g|^s \, P(g) \quad \textit{for all} \quad g \in L_n \,, \tag{32}$$

where $P(g)$ is a homogeneous T valued polynomial in the components of g (with respect to the canonical basis of $\mathbb{R}^n$), and s is some real number fixed by weight of the tensor density representation of $GL(n)$.

(c) *For a given $t \in \tilde{\mathcal{E}}_T$, we have*

$$t(g) = |\det g|^s \, P(g, \varepsilon(g)) \quad \textit{for all} \quad g \in L_n \,, \tag{33}$$

where $P(g, \varepsilon(g))$ is a homogeneous T valued polynomial in the components of g and the components[2] *of $\varepsilon(g) := \sqrt{\det g}\, \epsilon$ (with respect to the canonical basis of $\mathbb{R}^n$ in both cases), and s is some real number fixed by the weight of the tensor density representation of $GL^+(n)$.*

Remark 2.5. Since, in view of this Lemma, an equivariant tensor density $t(g)$ is a homogeneous function, say of degree k, of g up to a power of $|\deg g|$, it could always be rewritten as

$$t(g) = |\det g|^{\frac{k}{n}} \, t(|\det g|^{-\frac{1}{n}} \, g). \tag{34}$$

This observation will be later useful in the proof of Theorem 3.2.

[2]The homogeneous degree of $P(g, \varepsilon(g))$ counts the components of g with degree 2 and the components of $\varepsilon(g)$ with degree n.

Proof. We deal with the $GL(n)$-equivariant case, the $GL^+(n)$-equivariant case being completely analogous. The action of $GL(n)$ on both L_n and T is linear, so we denote it as $u \cdot x$, for $u \in GL(n)$ and x in either L_n or T.

The first crucial observation, as L_n consists of a single orbit of $GL(n)$, is that equivariance allows us to fully fix $t\colon L_n \to T$ provided that we know its value on $\eta \in L_n$, by the formula

$$t(g) = t(u_g \cdot \eta) = u_g \cdot t(\eta), \tag{35}$$

for any $g \in L_n$ and $u_g \in GL(n)$ such that $g = u_g \cdot \eta$. The second crucial observation is that, to make sure that the values of t are assigned consistently, $t(\eta)$ must be invariant under the isotropy subgroup of η, namely $O(1, n-1)$. In other words, $t(\eta)$ must belong to $\mathcal{I}_T$, with respect to the induced representation of $O(1, n-1)$ on T. The formula (35) clearly defines mutually inverse maps $\mathcal{E}_T \to \mathcal{I}_T$ and $\mathcal{I}_T \to \mathcal{E}_T$, thus establishing the isomorphism $\mathcal{E}_T \cong \mathcal{I}_T$ claimed in part (a).

Let us now prove part (b). Fix an (equivariant) embedding $\alpha\colon T \to M_n^p$. Since $t(\eta)$ is an element of $\mathcal{I}_T$, from the characterization of isotropic tensors in Proposition 2.7, it must be of the form

$$t(\eta) = \alpha^{-1}\left(\sum_{\sigma\in S_p} c^\sigma \eta_\sigma\right), \tag{36}$$

where c^σ are some scalar coefficients. Then for any $g \in L_n$ and a corresponding $u_g \in GL(n)$ such that $g = u_g \cdot \eta$,

$$\begin{aligned} t(g) = u_g \cdot t(\eta) &= \alpha^{-1}\left(|\det u_g|^r \sum_{\sigma\in S_p} c^\sigma (u_g \cdot \eta_\sigma)\right) \\ &= |\det g|^{r/2}\, \alpha^{-1}\left(\sum_{\sigma\in S_p} c^\sigma g_\sigma\right), \end{aligned} \tag{37}$$

where r is the density weight of the representation T and where we have used the notation

$$g_\sigma = g_{\sigma(i_1)\sigma(i_2)} \cdots g_{\sigma(i_{p-1})\sigma(i_p)} \tag{38}$$

for the corresponding monomial on L_n in terms of the components of g with respect to the canonical basis on $\mathbb{R}^n$. Clearly, the above formula can be rewritten as $t(g) = |\det g|^s\, P(g)$, with $s := r/2$. We observe that, from (37), that P is an homogeneous polynomial (of degree $p/2$) in the components of the metric, completing the proof of part (b).

The proof of (c) is strictly analogous, taking into account the identity $u_g \cdot \epsilon = \varepsilon(g)$, for any $u_g \in GL^+(n)$ such that, $u_g \cdot \eta = g$. □

3 Characterization of finite renormalizations of Wick polynomials

We generalize the discussion of local covariant fields from [11], where only metric dependence was allowed, to a more general context where other *background* *fields* are allowed in addition to the metric $\mathbf{g}$ on a spacetime M. In order to simplify the presentation, we will restrict the extra background fields to two scalar functions m^2 and ξ, which appear in the description of a scalar quantum field.

Generally speaking, background fields are described by sections $\mathbf{h}$ of suitable bundles $HM \to M$ over the manifolds M we consider. Covariance requires to deal with all such bundles *simultaneously* and *coherently*. In other words we deal with an assignment of a bundle $HM \to M$ to *every* manifold M and require that any embedding $\chi\colon M \to M'$ must give rise to a corresponding well-defined pullback map $\chi^*\colon \Gamma(HM') \to \Gamma(HM)$. This picture can be phrased properly with the language of category theory by means of the notion of *natural bundle*.

A **natural bundle** is a functor $H\colon \mathfrak{Man} \to \mathfrak{Bndl}$ from the category of smooth manifolds (where objects are connected, have fixed dimension n and morphisms are embeddings, which are necessarily local diffeomorphism) to the category of smooth bundles, such that a morphism $\chi\colon M \to M'$ induces a morphism $H\chi\colon HM \to HM'$ that is necessarily a bundle map (i.e., fiber preserving) and itself a local diffeomorphism. The required pullback $\chi^*\colon \Gamma(HM') \to \Gamma(HM)$ is then implicitly defined by $\mathbf{h}' \circ \chi = H\chi \circ (\chi^*\mathbf{h}')$, when $\mathbf{h}' \in \Gamma(HM')$.

One elementary example of a natural bundle is the functor $M \mapsto \mathbb{R} \times M$, the trivial scalar bundle, whose sections we call scalar fields. Another relevant example is $M \mapsto \mathring{S}^2T^*M$, the bundle of Lorentzian metrics; we will denote a section of $\mathring{S}^2T^*M \to M$ by $\mathbf{g}$. Other examples are are $M \mapsto T^*M$ and $M \mapsto \Lambda^2M$, the cotangent bundle and the bundle of 2-forms, whose sections could be interpreted as background electromagnetic fields, in the vector potential or field strength forms.

Remark 3.1. In the rest of the paper, focussing on the theory of a *real quantum scalar field*, φ, we make a more precise choice of the natural functor H. We suppose that the manifolds of the category $\mathfrak{Man}$ are connected, n-dimensional (for a fixed $n \geq 2$), and the functor H assigns $M \mapsto HM = \mathring{S}^2T^*M \times \mathbb{R} \times \mathbb{R}$, with a morphism $\chi\colon M \to M'$ inducing the standard tensor push-forward $H\chi = \chi_*\colon HM \to HM'$. Then, the sections $M \to HM$ are triples $\mathbf{h} = (\mathbf{g}, m^2, \xi)$ always consisting of:

(a) a *Lorentzian metric*, $\mathbf{g}$, making $(M, \mathbf{g})$ a *(smooth) Lorentzian spacetime* of fixed dimension $n \geq 2$,

(b) the pair of real scalar fields m^2 and ξ over M, with the respective physical meaning of the *squared mass* of the scalar field and a factor describing the *coupling with the scalar curvature*.

We stress that, exactly as in [11], we assume that the *parameters* m^2 and ξ are actually *functions* on M. Quantum field theory in curved spacetime is well-defined for both constant or variable m^2 and ξ. There is of course no obstacle in restricting them to constant functions, as we note in Remark 3.5. Moreover, as in [11], m^2 and ξ are allowed to have any real value.

Definition 3.1. Let us fix the natural bundle $H\colon \mathfrak{Man} \to \mathfrak{Bndl}$ as in Remark 3.1. A **background field** is a section $\mathbf{h}\colon M \to HM$ and we call the pair $(M, \mathbf{h})$ a **background geometry**, provided $\mathbf{h} = (\mathbf{g}, m^2, \xi)$ is such that $(M, \mathbf{g})$ is a time-orientable globally hyperbolic spacetime. Furthermore we define the following categories.

(a) $\mathfrak{BkgG}$ is the **category of background geometries**, having time-oriented background geometries as objects and morphisms given by smooth embeddings $\chi\colon M \to M'$ that preserve the background fields, $\chi^*\mathbf{h} = \mathbf{h}'$ on M', the time orientation, and causality, meaning that every causal curve between $\chi(p)$ and $\chi(q)$ in M' is the χ-image of a causal curve between p and q in M.

(b) $\mathfrak{BkgG}^+$ is the **category of oriented background geometries** having oriented and time-oriented background geometries as objects and morphisms as in $\mathfrak{BkgG}$, but also required to preserve the spacetime orientation.

To describe (off-shell) algebras of observables on background geometries, we need the notion of a *net of algebras* (or *pre-cosheaf of algebras*).

Definition 3.2. A **net of algebras** is an assignment of a unital $*$-algebra $\mathcal{W}(M, \mathbf{h})$ for every background geometry $(M, \mathbf{h})$ in $\mathfrak{BkgG}$ together with an assignment of an injective unital $*$-algebra homomorphism $\iota_\chi\colon \mathcal{W}(M, \mathbf{h}) \to \mathcal{W}(M', \mathbf{h}')$ for every morphism in $\mathfrak{BkgG}$. In other words $\mathcal{W}\colon \mathfrak{BkgG} \to \mathfrak{Alg}$ is a functor from background geometries category into the category of (complex) unital $*$-algebras whose morphisms are injective unital $*$-algebra homomorphism. We refer to a functor $\mathcal{W}\colon \mathfrak{BkgG}^+ \to \mathfrak{Alg}$ as a net of algebras as well.

For a fixed background geometry $(M, \mathbf{h})$, a **scalar quantum field** Φ is a $\mathcal{W}(M, \mathbf{h})$-valued distribution[3] $\Phi\colon \mathcal{D}(M) \to \mathcal{W}(M, \mathbf{h})$, where $\mathcal{D}(M)$ is the space of (complex valued, smooth, compactly supported) test functions[4] on M. For convenience of notation, we sometimes write $\Phi(f)$ for the smearing

$$\Phi(f) = \int_M \varphi(x) f(x)\, dg(x) \tag{39}$$

where $dg(x)$ is the volume form induced by the metric $\mathbf{g}$.

We are in a position to state our definition of a *locally covariant* field which extends and generalizes [11, Def.3.2].

Definition 3.3. A **locally covariant scalar quantum field** Φ is an assignment of a scalar quantum field $\Phi_{(M,\mathbf{h})}$ to each background geometry $(M, \mathbf{h})$ that satisfies the following identity for each morphism $\chi\colon (M', \mathbf{h}' = \chi^*\mathbf{h}) \to (M, \mathbf{h})$:

$$\iota_\chi(\Phi_{(M,\chi^*\mathbf{h})}(f)) = \Phi_{(M,\mathbf{h})}(\chi_* f)\,, \quad \text{for any } f \in \mathcal{D}(M'). \tag{40}$$

In other words, Φ is a natural transformation $\Phi\colon \mathcal{D} \to \mathcal{W}$ between the functors of test functions and algebras of observables.

In view of the penultimate remark in the definition, a locally covariant scalar quantum field could alternatively be called *natural* scalar quantum field.

Remark 3.2.

(1) In [11], $\mathbf{h}$ is nothing but the Lorentzian metric of the spacetime and the parameters m^2 and ξ appearing in the definition of the quantum fields generated by KG fields are considered external parameters. Here instead

[3]For every Hadamard quasifree state ω over $W(M, \mathbf{g})$ the map $\mathcal{D}(M) \ni f \mapsto \omega(\Phi(f))$ is a distribution in the proper sense. A weaker requirement allowing to smear fields with distributions of a suitable wavefront set can be given exploiting the so called Hörmander pseudotopology [11], but it is irrelevant for this work.

[4]Note that $\mathcal{D}\colon \mathfrak{Man} \to \mathfrak{LCV}$ is itself a (covariant) functor from manifolds to locally convex topological vector spaces. It maps a morphism $\chi\colon M \to M'$ to the induced *extension by zero* map $\chi_*\colon \mathcal{D}(M) \to \mathcal{D}(M')$.

we explicitly include them in $\mathbf{h}$. It is very easy to prove that the concrete locally covariant quantum fields appearing in [11] (scalar KG field and associated Wick polynomials, time-ordered Wick polynomials and their derivatives) satisfy our more general definition locally covariant quantum fields.

(2) Definition 3.2 includes two distinct though related notions: *locality* and *covariance*, both illustrated by the condition (40). Locality corresponds to the case where χ describes the inclusion $\chi\colon M \subset M'$, while covariance corresponds to an arbitrary allowed χ.

A Klein-Gordon scalar field φ, with mass m^2 and coupled with the scalar curvature through the constant ξ, is the most elementary, but non trivial, example of a locally covariant quantum field. *Wick polynomials* φ^k, $k = 2, 3, \ldots$, are special kinds of locally covariant scalar fields, *not uniquely* associated to a Klein-Gordon scalar field $\varphi =: \varphi^1$, whose detailed properties are described in [11, Sec.4]. Existence of such families of objects was established in [11]. The lack of uniqueness is physically interpreted as the existence of some remaining degrees of freedom in the renormalization procedure of Wick polynomials. The key result on finite renormalizations of Wick polynomials is

Proposition 3.1 ([11, Thm.5.1])**.** *If $\{\tilde{\varphi}^k\}$ and $\{\varphi^k\}$ are two families of Wick polynomial fields ($k \in \mathbb{N}$) of the same Klein-Gordon field $\varphi = \varphi^1 = \tilde{\varphi}^1$, then, for every fixed spacetime $(M, \mathbf{g})$, their difference can be parametrized as follows:*

$$\tilde{\varphi}^k_{(M,\mathbf{g})}(x) = \varphi^k_{(M,\mathbf{g})}(x) + \sum_{i=0}^{k-2} \binom{k}{i} C_{k-i}(x) \varphi^i_{(M,\mathbf{g})}(x), \tag{41}$$

where the

$$C_k(x) = C_k[\mathbf{g}^{ab}(x), R_{abcd}(x), \ldots, \nabla_{(e_1} \cdots \nabla_{e_{k-2})} R_{abcd}(x), \xi, m^2] \tag{42}$$

are some polynomials scalars tensorially formed from all of their arguments (except ξ) with coefficients that depend analytically on ξ, which scale as $C_k \mapsto \lambda^k C_k$ when their arguments are rescaled as $\xi \mapsto \xi$, $m^2 \mapsto \lambda^2 m^2$, $\mathbf{g}^{ab} \mapsto \lambda^2 \mathbf{g}^{ab}$ and $R_{abcd} \mapsto \lambda^{-2} R_{abcd}$, with the same scaling weight for its derivatives.

Remark 3.3 (Wick powers). We shall not enter into the details of the defining properties of Wick polynomials. We just list them with some comments, details can be found in Sec.4 of [11].

(i) **Locality and Covariance.** Each Wick power is a locally covariant quantum field, that is, Definition 3.3 is satisfied with $\Phi = \varphi^k$ for every k.

(ii) **Specific (algebraic and microlocal).** Wick powers have properties heuristically known to hold: Hermiticity and a specific expressions for their commutator with a free field. Moreover $\omega(\varphi^k(x))$ is a distribution with empty wavefront set for every Hadamard quasifree state ω.

(iii) **Scaling.** There is a linear action of physical scalings on the vector space of locally covariant quantum fields (Equation (48) of [11]). The k-the Wick power $\varphi^k_{(M,\mathbf{h})}$ has almost homogeneously degree k under this action, as in Definition 2.2 or Section 2.4.

(iv) Continuity and Analyticity. Wick polynomials vary continuously under smooth variations of the background fields, with this dependence strengthened to analytic (both in a suitable technical sense explained in [11]) on analytic metrics and analytic background fields.

Remark 3.4 (Scaling). Our scaling condition, which uses Definition 2.2, is slightly weaker than Definition 4.2 of [11], but it will be sufficient for our purposes. The difference is in the notion of *order* of the logarithmic terms. The 'order' in Definition 2.2 refers only to the scaling properties. On the other hand, the 'order' of a quantum field used in [11, Def.4.2] refers the number of iterated commutations with φ needed to annihilate that field. An inductive argument (in k) shows that if a Wick power φ^k satisfied Definition 4.2 of [11], then the same Wick power satisfies also our Definition 2.2(b).

Implementation of Continuity and Analyticity requirements was a quite technically difficult task in [11] as stressed in the introduction. We would like to demonstrate that, to prove the statement of Proposition 3.1, the Continuity and Analyticity properties are unnecessary (possibly even follow from the other properties) and are subsumed by the already required Local and Covariance properties. Dropping Continuity and Analyticity, we can achieve essentially the same result written below into a more precise form:

Theorem 3.2. *Let $\{\tilde{\varphi}^k\}$ and $\{\varphi^k\}$ be two families of locally covariant quantum fields ($k \in \mathbb{N}$) in the sense of Definition 3.3 (referring either to the category $\mathfrak{BkgG}$ of non-oriented background geometries or $\mathfrak{BkgG}^+$ of oriented background geometries) that satisfy the additional Specific (microlocal and algebraic) and Scaling properties*[5] *with respect to the Klein-Gordon field $\varphi = \varphi^1 = \tilde{\varphi}^1$.*

(a) *If φ is defined with respect to the category $\mathfrak{BkgG}$, for every $(M, \mathbf{h})$, we have:*

$$\tilde{\varphi}^k_{(M,\mathbf{h})}(x) = \varphi^k_{(M,\mathbf{h})}(x) + \sum_{i=0}^{k-2} \binom{k}{i} C_{k-i}[\mathbf{h}](x)\varphi^i_{(M,\mathbf{h})}(x), \tag{43}$$

where the

$$\begin{aligned} C_k[\mathbf{h}](x) = C_k \big[&\mathbf{g}^{ab}(x), R_{abcd}(x), \ldots, \nabla_{e_1} \cdots \nabla_{e_h} R_{abcd}(x), \\ &\xi(x), \ldots, \nabla_{e_1} \cdots \nabla_{e_r}\xi(x), m(x)^2, \ldots, \nabla_{e_1} \cdots \nabla_{e_s} m(x)^2\big] \end{aligned} \tag{44}$$

are some scalar polynomials, tensorially formed from all of their arguments, except $\xi(x)$, and where $R_{abcd}(x)$ denotes the Riemann tensor and ∇_a the Levi-Civita connection of $\mathbf{g}_{ab}$ at $x \in M$.

(b) *If φ is defined with respect to the category $\mathfrak{BkgG}^+$, for every $(M, \mathbf{h})$, we have a variant of (43) with*

$$\begin{aligned} C_k[\mathbf{h}](x) = C_k \big[&\mathbf{g}^{ab}(x), \varepsilon^{a_1\cdots a_n}(x), R_{abcd}(x), \ldots, \nabla_{e_1} \cdots \nabla_{e_h} R_{abcd}(x), \\ &\xi(x), \ldots, \nabla_{e_1} \cdots \nabla_{e_r}\xi(x), m(x)^2, \ldots, \nabla_{e_1} \cdots \nabla_{e_s} m(x)^2\big] \end{aligned} \tag{45}$$

scalar polynomials, tensorially formed from all of their arguments, except $\xi(x)$, and now including the Levi-Civita tensor $\varepsilon^{a_1\cdots a_n}(x)$ of $\mathbf{g}_{ab}$ at $x \in M$.

[5] That is, they are Wick polynomials of φ in the sense of [11], up to the Continuity and Analyticity properties.

In both cases (a) and (b), the coefficients of the polynomials are smooth (instead of analytic) functions of $\xi(x)$ whose functional form does not depend on M.

Further, the C_k scale as $C_k \mapsto \lambda^k C_k$ when their arguments are rescaled as follows: $\xi \mapsto \xi$, $m^2 \mapsto \lambda^2 m^2$, $g^{ab} \mapsto \lambda^2 g^{ab}$, $\varepsilon^{a_1 \cdots a_n} \mapsto \lambda^n \varepsilon^{a_1 \cdots a_n}$, $R_{abcd}(x) \mapsto \lambda^{-2} R_{abcd}(x)$ and the covariant derivatives do not change this rescaling behaviour as the coordinates are dimensionless. These rescaling properties fix the order of the polynomial C_k.

Obviously all terms $\nabla_{e_1} \cdots \nabla_{e_s} \xi(x)$ and $\nabla_{e_1} \cdots \nabla_{e_s} m(x)^2$ with $s > 0$ vanish if, at the end of computation, m^2 and ξ are taken constant.

The proof of our main Theorem 3.2 will be mainly geometric. However, we will need an intermediate analytical result, which we encapsulate in the Lemma below, which is a more detailed version of the first two paragraphs of the proof of [11, Thm.5.1]. Logically, this analytical result follows from the Microlocal Spectrum property, form the Locality requirement (cf. (2) in Remark 3.2) and the from Scaling requirement. It does not rely on either the Continuity or Analyticity properties nor the Covariance requirement.

Lemma 3.3. *For $\{\tilde{\varphi}^k\}$ and $\{\varphi^k\}$ as in Theorem 3.2 and every fixed M, the identity (43) holds with some smooth functions $C_k[\mathbf{h}]$, where the value $C_k[\mathbf{h}](x)$ depends only on the germ of $\mathbf{h}$ at $x \in M$. Moreover, these functions are locally covariant, so that $\chi^* C_k[\mathbf{h}] = C_k[\chi^* \mathbf{h}]$ for any morphism χ in $\mathfrak{BkgG}$ (resp. $\mathfrak{BkgG}^+$), and $C_k[\mathbf{h}]$ scales almost homogeneously of degree k under the physical scaling transformation $\mathbf{h} = (\mathbf{g}, m^2, \xi) \mapsto (\lambda^{-2}\mathbf{g}, \lambda^2 m^2, \xi)$.*

Proof of Lemma 3.3. The proof is inductive in k. The thesis holds for $k = 1$ and $C_1 = 0$, since $\varphi^1 = \tilde{\varphi}^1$. Next suppose that (43) holds for some functions $C_i \colon \Gamma(HM) \to C^\infty(M)$, $i = 1, 2, \ldots, k-1$, that satisfy the desired properties. Then $C_i[\mathbf{h}]1$ defines a locally covariant c-number field, where $1 \in W(M, \mathbf{h})$ is the identity of the given algebra. Define

$$\Phi_{k,(M,\mathbf{h})}(x) := \tilde{\varphi}^k_{(M,\mathbf{h})}(x) - \left(\varphi^k_{(M,\mathbf{h})}(x) + \sum_{i=0}^{k-2} \binom{k}{i} C_{k-i}[\mathbf{h}](x) \varphi^i_{(M,\mathbf{h})}(x) \right). \quad (46)$$

By construction, Φ_k is a locally covariant quantum field as in Definition 3.3. It satisfies the requirements in Remark 3.3: *Specific* (microlocal and algebraic) and *Scaling* properties, as outlined in Remark 3.4. The *algebraic properties* in particular require that Φ_k is Hermitian and, on any given spacetime M, it satisfies $[\Phi_{k,(M,\mathbf{h})}(x), \varphi(y)] = 0$ for all $x, y \in M$, which means that it is a c-number from [11, Prop.2.1]. In other words, $\Phi_{k,(M,\mathbf{h})} = C_k[\mathbf{h}]1$ where $C_k[\mathbf{h}] : C_0^\infty(M) \to \mathbb{R}$ is a distribution. Sticking to Specific properties, the *microlocal spectrum condition* then implies that the distribution $C_k[\mathbf{h}]$ is a smooth function of x. The locality requirement of Definition 3.3 (see (2) in Remark 3.2) entails that $\chi^* C_k[\mathbf{h}] = C_k[\chi^* \mathbf{h}]$ for any inclusion $\chi \colon U \subset M$. In other words, fixing $x \in M$ and taking the limit over decreasing neighborhoods U of x, the value $C_k[\mathbf{h}](x)$ depends only on the germ of $\mathbf{h}$ at x.

The validity of the *Scaling* property for both φ^k and $\tilde{\varphi}^k$ imply that, by the formula (46), Φ_k is a linear combination of products of terms with almost homogeneous degrees that add up to k. Thus, by Lemma 2.5, Φ_k itself has almost homogeneous degree k and thus

$$S_\lambda \Phi_k = \lambda^k \Phi_k + \lambda^k \sum_i (\log^i \lambda) \Psi_i, \tag{47}$$

where S_λ is the action of physical scalings mentioned in Remark 3.4 and Ψ_i are some other locally covariant quantum fields of almost homogeneous degree k. Recall that, for a fixed spacetime M, we can think of a locally covariant field as a section of a certain bundle over $\Gamma(HM)$, where the fiber over $\mathbf{h} \in \Gamma(HM)$ is the space of $\mathcal{W}(M, \mathbf{h})$-valued distributions on M. Note also that, as discussed in Section 4.3 of [11], all the $\mathcal{W}(M, \mathbf{h})$ algebras are naturally isomorphic and the action S_λ consists of the composition of action induced from $\Gamma(HM)$ and the rescaled natural isomorphism. Thus, as Φ_k is a c-number, so are all of the Ψ_i. Hence, from the definition of S_λ, the action of physical scalings on $\Phi_k = C_k 1$ restricts to the induced action of physical scalings on $C^\infty(M)$ valued functions on $\Gamma(HM)$. Therefore, $C_k[\mathbf{h}]$ has almost homogeneous degree k as a function on $\Gamma(HM)$. □

In the proof of the main Theorem below, we systematically make use of the geometric results summarized in Section 2. In particular, the *non-linear Peetre theorem* discussed in Section 2.3 brings in the key simplification in our proof in comparison with the arguments of [11]. This theorem is well known in differential geometry but has not before been applied in this context. It states that, under the conditions given by Lemma 3.3 and the Locality property, the C_k must be some (possibly non-linear) differential operators of locally bounded order applied to the background fields $\mathbf{g}$, m^2 and ξ. It then remains only to call upon the Scaling and Covariance properties to check that the C_k may only be of the form stipulated in Equation (44) or (45).

Proof of Theorem 3.2. In this proof, we carefully separate the hypotheses of *locality*, *scaling* and *covariance*. Locality allows us to conclude that the functions C_k are differential operators. Scaling restricts their form and then covariance restricts their form even further, to the desired result. Note that, unlike in [11] we do not make use of Riemann normal coordinates. As a result, we invoke the transformations properties of C_k under two different kinds of scaling transformations, which are mixed when normal coordinates are employed.

1. Locality and the Peetre theorem. The first step is to combine the *locality* of the coefficients C_k of Equation (43) with the *non-linear Peetre theorem* (Proposition 2.2) to conclude that in fact these coefficients are *differential operators of locally bounded order* (see Section 2.3 for details). To verify the hypotheses of Proposition 2.2, take the bundle $F \cong \mathbb{R} \times M \to M$, so that its sections are just real valued functions $\Gamma(F \to M) = C^\infty(M)$. Finally, take the bundle $E \cong HM \to M$. Lemma 3.3 shows that $C_k \colon \Gamma(HM) \to C^\infty(M)$ such that $C_k[\mathbf{h}](x)$ depends only on the germ of $\mathbf{h}$ at $x \in M$. Consequently, the non-linear Peetre theorem gives us the desired result: for every fixed $M \in \mathfrak{Man}$, $C_k \colon \Gamma(HM) \to C^\infty(M)$ is a differential operator of locally bounded order, as defined in Section 2.3.

Although we treat m^2 and ξ as spacetime-dependent fields, this is not crucial. They could be treated as constant parameters from the start and the slight modification of the proof, needed only at this point, is discussed in Remark 3.5.

2. Almost homogeneity under physical scaling. Consider a Lorentzian metric $\mathbf{g}_0$ on M, as well as a point $y \in M$ and an open neighborhood U of y with compact closure, with a coordinate system (x^a) centered at y. Since C_k is a differential operator of locally bounded order, for any such $\mathbf{g}_0$, y and U there exists an integer $r \geq 0$ such that C_k is a differential operator on U of local order r when acting on sections of HM close to $(\mathbf{g}_0, m^2 = 0, \xi = 0)$, in a precise sense that we discuss next. Naturally, the coordinates x^a induce (scaling) adapted local coordinates on the jet bundle $J^r HM$, which we write as $(x^a, g, g_{ab}, g^{ab,A}, w^A, z^A)$, recalling that the coordinates (g, g_{ab}) are functionally independent up to the identity $|\det g_{ab}| = g$. The notation and the meaning of these coordinates are discussed in Section 2.2. The only difference is that we now use two sets of coordinates, w^A and z^A, for the jets of the scalar fields, $m^2(x)$ and $\xi(x)$ respectively, instead of just one. Then, by the bound r on the local order of C_k at y, there exists a neighborhood $V_1^r \subseteq J^r HM$ of $j^r_y(\mathbf{g}_0, m^2 = 0, \xi = 0)$, projecting onto U, and a function $F_k(x^a, g, g^{ab,A}, w^A, z^A)$ defined on V_1^r such that

$$C_k[\mathbf{h}](x) = F_k(j^r \mathbf{h}(x)), \tag{48}$$

for any section $\mathbf{h} \in \Gamma(HM|_U \to U)$ such that $j^r\mathbf{h}(U) \subseteq V_1^r$. Note that V_1^r may be strictly smaller than $J^r H|_U$. Without loss of generality, but possibly shrinking the domain of F_k, we can choose it such that $V_1^r \cong U \times W_1^r$, where the projection on the U factor is effected by the base coordinates (x^a) and the projection onto W_1^r is effected by the remaining fiber coordinates. The main obstacle to increasing V_1^r to all of $J^r HM$ is the possible need to increase the order r on larger domains. At the moment, from the Peetre theorem, we know only that the order r of C_k is locally bounded, but may not have a finite global bound. The subscript $_1$ on V_1^r will increase in the subsequent discussion as we use the properties of C_k to gradually enlarge the domain of definition of the function F_k, while maintaining the identity (48), and thus the bound r on the order of C_k. In the final step of the proof we will in fact show that differential order of C_k is actually globally bounded. With that in mind, it is then consistent, on a first reading of the proof, to assume that r is globally fixed and $V_1^r = J^r HM$, so that the parts dealing with enlarging V^r could be skipped.

Similar to Equation (20), the vector field implementing infinitesimal physical scaling transformations on $V_1^r \subseteq J^r HM$ is

$$e_1 = (2 + 2|A|)g^{ab,A}\partial_{ab,A} + (2 + 2|A|)w^A \partial^w_A + 2|A|z^A \partial^z_A. \tag{49}$$

According to the last statement in Lemma 3.3, the coefficient C_k and hence the function F_k scale almost homogeneously with degree k with respect to the vector field e_1 (Lemma 3.3). Therefore, according to Lemma 2.4, there exists an integer $l > 0$ and function H_j on V_1^r, for $j = 0, \dots, l - 1$, such that

$$F_k = g^{-\frac{k}{2n}} \sum_{j=0} \log^j(g^{-\frac{1}{2n}}) H_j, \tag{50}$$

where each H_j is invariant under the action of e_1 and hence can be written as

$$H = H(x^a, g^{-\frac{1}{n}} g_{ab}, g^{\frac{1}{2n} + \frac{1}{n}|A|} g^{ab,A}, g^{\frac{1}{n} + \frac{1}{n}|A|} w^A, g^{\frac{1}{n}|A|} z^A). \tag{51}$$

At this point, we may extend the domain V_1^r to $V_2^r \subseteq J^r HM$, which is invariant under physical scaling. That is, we can write $V_2^r \cong \mathbb{R}^+ \times W_2^r$, where the coordinate g effects the projection onto the $\mathbb{R}^+$ factor and the coordinates $(x^a, g^{-\frac{1}{n}} g_{ab}, g^{\frac{1}{2n}+\frac{1}{n}|A|} g^{ab,A}, g^{\frac{1}{n}+\frac{1}{n}|A|} w^A, g^{\frac{1}{n}|A|} z^A)$ effect the projection onto the W_2^r factor, which includes at least the point $(g^{-\frac{1}{n}} g_{ab} \circ \mathbf{g}_0(y), 0, 0, 0)$. The function F_k extends from V_1^r to V_2^r in a unique way as an almost homogeneous function of degree k.

Let us go into some of the details of the mentioned unique extension procedure. So far, we could only presume that the identity (48) that expresses the function $C_k[\mathbf{h}](x)$ in terms of the differential operator defined by the function F_k holds only when the germ of $\mathbf{h}$ at $x \in M$ projects onto one of the jets in the domain $V_1^r \subseteq J^r HM$ of F_k. We have defined the extended domain V_2^r to be the smallest domain invariant under physical scaling and containing V_1^r. The function F_k, by using formula (50), has a unique almost homogeneous extension to V_2^r that scales almost homogeneously and agrees with the known values of F_k on V_1^r. Since any element of V_2^r can be brought back to V_1^r by a physical scaling transformation and $C_k[\mathbf{h}]$ itself scales almost homogeneously, the identity (48) must remain valid also for germs of $\mathbf{h}$ at x that project to jets in the extended domain V_2^r. Below, we use similar logic each time the domain of the function F_k is expanded, eventually to all of $J^r HM$, though possibly with a larger value of r, thus showing that $C_k[\mathbf{h}]$ is actually a differential operator of globally bounded order.

3. Diffeomorphism covariance and the Thomas replacement theorem. Now we move on to the *covariance property* of the C_k under diffeomorphisms, which will be used in several stages. First, part of the almost homogeneous scaling property implies that the functions H_j from Equation 51 are each separately invariant under diffeomorphisms, in the sense described in Section 2.5. Therefore, applying Proposition 2.6 we can conclude that

$$\begin{aligned} g^{-\frac{k}{2n}} H_j(x^a, g^{-\frac{1}{n}} g_{ab}, g^{\frac{1}{n}+\frac{1}{n}|A|} g^{ab,A}, g^{\frac{1}{n}+\frac{1}{n}|A|} w^A, g^{\frac{1}{n}|A|} z^A) \\ = g^{-\frac{k}{2n}} G_j(g^{-\frac{1}{n}} g_{ab}, g^{\frac{3}{n}+\frac{1}{n}|A|} \bar{S}^{ab(cd,A)}, g^{\frac{1}{n}+\frac{1}{n}|A|} \bar{w}^A, g^{\frac{1}{n}|A|} \bar{z}^A), \quad (52) \end{aligned}$$

where the notation use for the coordinates is explained in Section 2.5 and each $g^{-\frac{k}{2n}} G_j$, for $j = 0, \ldots, l-1$, is invariant under the natural action of either $GL(n)$ (or $GL^+(n)$, depending on which of the cases (a) or (b) we are dealing with) on its arguments. Notably, G_j depends neither on the base (x^a) nor on the Christoffel coordinates $(\Gamma^a_{(bc,A)})$. Since the overall defined function C_k is invariant under diffeomorphisms, the domain V_2^r where the identity (48) holds can now be extended to $V_3^r \subset J^r HM$ that is invariant under the action of diffeomorphisms. More precisely, we have $V_3^r \cong U \times L_n \times \mathbb{R}^\gamma \times W_3^r$, where the coordinates (x^a) effect the projection onto the U factor, the coordinates (g_{ab}) or $(g, g^{-\frac{1}{n}} g_{ab})$ effect the projection onto the L_n factor (the whole space of non-degenerate bilinear forms on $\mathbb{R}^n$ with Lorentzian signature), the coordinates $\Gamma^a_{(bc,A)}$ effect the projection on the $\mathbb{R}^\gamma$ argument and the remaining coordinates $(g^{\bar{3}n+\frac{1}{n}|A|} \bar{S}^{ab(cd,A)}, g^{\frac{1}{n}+\frac{1}{n}|A|} \bar{w}^A, g^{\frac{1}{n}+\frac{1}{n}|A|} \bar{z}^A)$ effect the projection on the W_3^r argument, which contains at least the point $(0, 0, 0)$ and is invariant under the corresponding action of $GL(n)$ (resp. $GL^+(n)$).

4. Invariance under coordinate scaling. Next, recall the action of the subgroup of $GL(n)$ (resp. $GL^+(n)$) that we called *coordinate scalings* in Section 2.5. Notice that all the coordinates that the functions G_j depend on have positive weight with respect to coordinate scalings, with the exception of $(g^{-\frac{1}{n}} g_{ab}, z)$.

For brevity, let us rewrite our coordinates as $(g, g^{-\frac{1}{n}} g_{ab}, z, q^i)$, with the weight of the coordinate q^i under coordinate scalings denoted by $d_i > 0$. Then the invariance of the functions F_k on V_3^r under diffeomorphisms, and hence coordinate scalings, implies the identity

$$\begin{aligned} \mu^k F_k(g, g^{-\frac{1}{n}} g_{ab}, z, q^i) &= \mu^k F_k(\mu^{2n} g, g^{-\frac{1}{n}} g_{ab}, z, \mu^{d_i} q^i) \\ &= g^{-\frac{k}{2n}} \sum_{j=0}^{l-1} \log^j(\mu g^{-\frac{1}{2n}}) G_j(g^{-\frac{1}{n}} g_{ab}, z, \mu^{d_i} q^i) \end{aligned} \tag{53}$$

for any point of V_3^r on its left hand side and any value of $\mu > 0$. As described above, the limit $(g^{-\frac{1}{n}} g_{ab}, z, 0)$ of the arguments of the functions G_j as $\mu \to 0$ falls within the domain of the functions G_j. Therefore, while the limit of the left-hand side of (53) converges to 0 as $\mu \to 0$, the right-hand side diverges unless all $G_j = 0$ for $j > 0$, so that $F_k = g^{-\frac{k}{2n}} G_0$. The new identity implied by invariance under coordinate scalings is then

$$g^{-\frac{k}{2n}} G_0(g^{-\frac{1}{n}} g_{ab}, z, q^i) = \mu^{-k} g^{-\frac{k}{2n}} G_0(g^{-\frac{1}{n}} g_{ab}, z, \mu^{d_i} q^i). \tag{54}$$

Fix some values for the coordinates $(g, g^{-\frac{1}{n}} g_{ab}, z)$ and recall that the point $(g^{-\frac{1}{n}} g_{ab}, z, 0)$ is part of the domain of definition of G_0. Since G_0 is smooth, Taylor's theorem allows us to write it as

$$G_0(g^{-\frac{1}{n}} g_{ab}, z, q^i) = \sum_{|I|<N} A_I(g^{-\frac{1}{n}} g_{ab}, z) q^I + O(q^N), \tag{55}$$

where $I = i_1 \cdots i_m$ is a multi-index with respect to the coordinates (q^i) and $N > 0$ is an integer large enough so that $\langle d, I \rangle = \sum_{j=1}^m d_{i_j} > k$ for any $m = |I| > N$. Note that the error term $O(q^N)$, for fixed (q^i) mapped to $(\mu^{d_i} q^i)$ and $\mu \to 0$, is mapped to $O(\mu^{k+1})$ by our choice of sufficiently large N. Thus, using Taylor's theorem, we can rewrite (54) as

$$g^{-\frac{k}{2n}} G_0(g^{-\frac{1}{n}} g_{ab}, z, q^i) = \sum_{|I|<N} g^{-\frac{k}{2n}} A_I(g^{-\frac{1}{n}} g_{ab}, z) q^I \mu^{\langle d, I \rangle - k} + \mu^{-k} O(\mu^{k+1}). \tag{56}$$

While the left-hand side of (56) is bounded as $\mu \to 0$, the right-hand side diverges unless all $A_I = 0$ for I such that $\langle d, I \rangle < k$. If this vanishing condition is satisfied, the $\mu \to 0$ limits of both sides of (56) exist and give the identity

$$F_k = g^{-\frac{k}{2n}} G_0(g^{-\frac{1}{n}} g_{ab}, z, q^i) = \sum_{\langle d, I \rangle = k} g^{-\frac{k}{2n}} A_I(g^{-\frac{1}{n}} g_{ab}, z) q^I. \tag{57}$$

At this point, we can once more enlarge the domain of definition of the function F_k, where the identity (48) holds, from V_3^r to $V_4^r \subset J^r HM$. The new domain is isomorphic to $V_4^r \cong U \times L_n \times \mathbb{R} \times W_4 \times \mathbb{R}^\gamma \times \mathbb{R}^\delta$, where the coordinates (x^a) effect the projection onto the U factor, the coordinates (g_{ab}) or $(g, g^{-\frac{1}{n}} g_{ab})$ effect the projection onto the L_n factor, the coordinate $(g^{\frac{1}{n}} w, z)$ effects the projection onto the $\mathbb{R} \times W_4$ factor (which at least contains the point $(0,0)$), the coordinates $(\Gamma^a_{(bc,A)})$ effect the projection onto the $\mathbb{R}^\gamma$ factor, and the remaining coordinates $(g^{\frac{3}{n}+\frac{1}{n}|A|} \bar{S}^{ab(cd,A)}, g^{\frac{1}{n}+\frac{1}{n}|A|} \bar{w}^A, g^{\frac{1}{n}+\frac{1}{n}|A|} \bar{z}^a)$ effect the projection onto the $\mathbb{R}^\delta$ factor. Note that $U \times L_n \times \mathbb{R} \times W_4 \subseteq HM$ and that V_4^r is simply its preimage with respect to the bundle projection $J^r HM \to HM$. The function F_k extends uniquely from V_3^r to a function on V_4^r that is invariant under coordinate scalings. The reason we could extend the domain so much is because almost all coordinates had positive degrees with respect to coordinate scalings. The range of the (z) coordinate is limited to W_4 because it is invariant under coordinate scalings and even under the larger group $GL(n)$ (resp. $GL^+(n)$) that acts on the other bundle coordinates.

5. $GL(n)$-equivariance and polynomial dependence on the metric. From the preceding discussion, the function F_k, satisfying the identity (48), depends only on the coordinates corresponding to the factors $V_4 = L_n \times W_4 \times \mathbb{R}^\delta$. Moreover, the dependence on the coordinates on the $\mathbb{R}^\delta$ factor is polynomial, while the coefficients $g^{-\frac{k}{2n}} A_I(g^{-\frac{1}{n}} g_{ab}, z)$ of these polynomials depend only on the $L_n \times W_4$ factor. It is also clear from the preceding discussion that each of the factors in V_4 carries a tensor density representation of $GL(n)$ (resp. $GL^+(n)$) (cf. Section 2.6), which happens to be trivial on W_4. The space of functions on V_4 then itself carries a representation of $GL(n)$ (resp. $GL^+(n)$), induced by the pullback of the action on V_4, and the function F_k is invariant under this action. In the same way, the space $\mathcal{P}_\delta^N$ of polynomials of degree no greater than N on $\mathbb{R}^\delta$ carries a representation of $GL(n)$ (resp. $GL^+(n)$),

$$(uP)(\rho) = P(u^{-1}\rho), \quad \text{for any } u \in GL(n),\ P \in \mathcal{P}_\delta^N \text{ and } \rho \in \mathbb{R}^\delta, \tag{58}$$

which by elementary reasoning, within the representation theory of $GL(n)$ [7], is a direct sum of tensor density representations. Let us group these subrepresentations by tensor rank and density weight. Therefore, $\mathcal{P}_\delta^N = \bigoplus_j T_j$, where each T_j is a tensor density representation.

The form that we have reduced F_k to can be described as follows. Given a point $(\mathbf{g}, \xi, \rho) \in V_4$, the *A-coefficients* $g^{-\frac{k}{2n}} A_I(g^{-\frac{1}{n}} g_{ab}, z)$ evaluated at $(\mathbf{g}, \xi) \in L_n \times W_4$ give a polynomial in $\mathcal{P}_\delta^N$, which is then evaluated at $\rho \in \mathbb{R}^\delta$. Thus we can think of the A-coefficients as a collection of functions $A_j \colon L_n \times W_4 \to T_j$, with components given by

$$(A_j(g_{ab}, z))_I = g^{-\frac{k}{2n}} A_I(g^{-\frac{1}{n}} g_{ab}, z). \tag{59}$$

The only way for F_k constructed in this way to be invariant under the action of $GL(n)$ is for the maps A_j to be equivariant (cf. Section 2.6), so that

$$\begin{aligned} F_k(u\mathbf{g}, u\xi, u\rho) &= \sum_j A_j(u\mathbf{g}, u\xi)(u\rho) = \sum_j (uA_j(\mathbf{g}, \xi))(u\rho) \\ &= \sum_j A_j(\mathbf{g}, \xi)(u^{-1}u\rho) = \sum_j A_j(\mathbf{g}, \xi)(\rho) = F_k(\mathbf{g}, \xi, \rho), \end{aligned} \tag{60}$$

for any $u \in GL(n)$ (resp. $GL^+(n)$) and $(\mathbf{g}, \xi, \rho) \in V_4$.

We are finally in a position to conclude that, for a fixed $\xi \in W_4$, the map $A_j(-, \xi) \colon L_n \to T_j$ is an *equivariant tensor density*, in the sense of Definition 2.6, and hence must be of the form dictated by Lemma 2.8, which characterizes all such maps in a way, in view of Remark 2.5, compatible with our formula (59). In other words, the coefficients of the polynomials $A_j(\mathbf{g}, \xi)$ depend themselves polynomially on the components g_{ab} and $\varepsilon_{a_1 \cdots a_n}$ of the covariant metric and

Levi-Civita tensors, up to an overall multiple of $g = |\det g_{ab}|$. If F_k is invariant under $GL(n)$, then the dependence on $\varepsilon_{a_1\cdots a_n}$ must be trivial, while it could in general be non-trivial if F_k is invariant only under $GL^+(n)$. Expanding all the polynomials in g_{ab}, $\varepsilon_{a_1\cdots a_n}$ and q^i, all the factors of powers of g must collectively cancel to preserve invariance of F_k under $GL(n)$ (resp. $GL^+(n)$). In other words, we can conclude that

$$F_k = \sum_j a_j(z) P_j(g_{ab}, \varepsilon_{a_1\cdots a_n}, \bar{S}^{ab(cd,A)}, \bar{w}^A, \bar{z}^A), \tag{61}$$

where the sum is over a (necessarily finite) basis of polynomials P_j, which consist of linear combinations of tensor contractions of products of their arguments, with coefficients arbitrarily depending on the z coordinate. In this form, the function F_k is manifestly invariant under $GL(n)$ (resp. $GL^+(n)$) transformations.

6. Global boundedness of differential order. To conclude the proof, it remains only to extend the domain V_4^r once more, this time to all of $J^r HM$, for an appropriate choice of r. It is well known that for a fixed weight k under coordinate scaling, there is only a finite number of linearly independent polynomials P_j constructed, as described above, from the metric and the covariant derivatives of the scalar fields m^2, ξ and the Riemann curvature tensor in the form S^{abcd}, even if the number of the derivatives r is allowed to be arbitrary [6]. Let r_k be the maximum number of derivatives that appear in a basis for these polynomials P_j. Then, no matter the original choice of domain $U \subseteq M$, the differential operator C_k restricted to it must be of order $\leq r_k$. Thus, we are justified in setting $r = r_k$ in all of the preceding discussion. The only obstacle that may have prevented us from extending the domain $V_4^r \subseteq J^{r_k} HM$ of the function F_k to all of the pre-image of U under the projection $J^{r_k} HM \to M$ is the possibility that C_k would change order on jets whose projections fall outside $V_4^{r_k}$. However, with the maximal possible order of C_k bounded by r_k, this obstacle is now absent. In other words, we can safely presume that $V_4^{r_k}$ is equal to the preimage of $U \subseteq M$ with respect to the projection $J^{r_k} HM \to M$, with F_k retaining the form (61) on all of its domain. Finally, covariance of C_k with respect to diffeomorphisms requires that the form (61) is also independent of the domain $U \subseteq M$. Thus, we can conclude that there exists a globally defined smooth bundle map $F_k \colon J^{r_k} HM \to \mathbb{R} \times M$ over M of the form (61) such that $C_k[\mathbf{h}](x) = F_k \circ j^{r_k}\mathbf{h}(x)$ for any $x \in M$ and $\mathbf{h} \in \Gamma(HM)$, which concludes the proof. □

Remark 3.5. We observe, by looking at the *Locality and the Peetre theorem* step of the above proof and also at the proof of Lemma 3.3, that one might wonder at the need to take m^2 and ξ as spacetime-dependent fields rather than constants, as is usually the case. Our arguments still go through, with only two changes. First, the microlocal hypothesis mentioned in 3.3 must be strengthened to require an empty wavefront set for $\omega(\varphi^k(x))$ as a distribution on $M \times \mathbb{R}^2$ (with the $\mathbb{R}^2$ factor standing for the parameter space of m^2 and ξ) rather than as a distribution on M for any fixed m^2 and ξ. Note that the weaker microlocal requirement does not exclude the infinite family of counterterms of [20] that were discussed in the Introduction, while the stronger one does. Second, we must make use of the more general version of the Peetre theorem for differential operators with parameters, as in Proposition A.1 in Appendix A. To apply that result, we would need to let $N = M$ and replace the spacetime manifold M by $P = M \times \mathbb{R}^2$, adding the (m^2, ξ) parameter space. It would then follow from known information about C_k that it is local with respect to the natural projection $P \cong \mathbb{R}^2 \times M \to M$, hence satisfying the more general Peetre theorem.

We end this section with a couple of straight forward but noteworthy observations. First, it is a direct result of the proof of Lemma 3.3 that the set of coefficients $\{C_k[\mathbf{h}]\}$ from Equation (43) is determined jointly by the entire families $\{\varphi^k\}$ and $\{\tilde{\varphi}^k\}$ of Wick powers, rather than depending on each pair φ^k and $\tilde{\varphi}^k$ individually. Second, the converse of Theorem 3.2 holds as well. That is, given a family $\{\varphi^k\}$ of locally covariant Wick powers and a set $\{C_k[\mathbf{h}]\}$ of satisfying the conclusions of Theorem 3.2, the formula (43) defines another family $\{\varphi^k\}$ of locally covariant Wick powers.

4 Discussion

In this work, we have characterized admissible finite renormalizations of Wick polynomials of a locally covariant quantum scalar field φ on curved spacetimes, with possibly spacetime-dependent mass m^2 and curvature coupling ξ. By local covariance, we mean the axioms of Brunetti, Fredenhagen and Verch [5]. Our work is a significant technical improvement on the original work of Hollands and Wald [11] on this subject. The main result (Theorem 3.2) is a slight generalization of that of Hollands and Wald, yet our hypotheses are significantly weaker and the proof is greatly simplified and streamlined.

Under standard hypotheses, on Minkowski space, where the curvature coupling ξ is absent, it is well known that the finite renormalizations of the Wick polynomial φ^k are restricted to linear combinations of Wick polynomials of lower order, with dimensionful coefficients that are polynomials in m^2, with the total dimension matching that of φ^k. This is a strong constraint, because the resulting space of possibilities is finite-dimensional. On curved spacetimes, as first proven by Hollands and Wald in [11], adding local covariance and some further more technical hypotheses gives a result of comparable strength. The only modification is that the coefficients of lower order Wick polynomials can also depend polynomially on curvature scalars and analytically on ξ, with the same restriction on their dimensions. The resulting possibilities no longer form a finite-dimensional space, but a quasi-finite-dimensional one, in the sense that it is finitely generated under linear combinations with coefficients analytic in ξ. It is worth noting that the dependence of finite renormalization terms on the background metric is entirely contained in the curvature scalars, while their ξ-dependent coefficients must be assigned uniformly across all spacetimes to preserve local covariance.

The hypotheses of Hollands and Wald, briefly recalled in Remark 3.3, include the requirements of *locality* and of *continuous* and *analytic* dependence on the background spacetime metric and coupling parameters. Unfortunately, while playing a crucial role in the existing proof, the analytic dependence hypothesis has been long considered somewhat unnatural and technically very cumbersome. We have found that, by using a standard result of differential geometry (the non-linear Peetre theorem, cf. Proposition 2.2 and Appendix A), in the presence of the remaining assumptions, the role of *both* the continuity and analyticity hypotheses is complete subsumed by that of locality. Thus, despite weakening our hypotheses by removing the continuity and analyticity requirements, our final result on the characterization of finite renormalizations of Wick polynomials, as stated in Theorem 3.2, is essentially identical to that of Hollands and Wald. The main difference is that arbitrary smooth dependence on the coupling ξ is now allowed, instead of just analytic dependence. Another difference is that we have also explicitly considered weakening covariance to only under orientation preserving diffeomorphisms, which increases the renormalization freedom to curvature scalars constructed also with the Levi-Civita tensor and not just the metric. Finally, we explicitly treat m^2 and ξ as possibly spacetime-dependent parameters, rather than simple constants. The original proof of Hollands and Wald also treated them as spacetime-dependent, while restricting to the case of constants in the statement of their final result. We noted in Remark 3.5 how our arguments could be adapted to treating the parameters as constants throughout.

As was already mentioned, our characterization of finite renormalizations extends to theories that need only be covariant with respect to orientation preserving diffeomorphisms. In particular, in even dimensions, chiral theories (those not invariant under spatial parity transformations) could be admissible. While our result does not contain any surprises, it is important to have a rigorous statement on the complete range of possibilities. In particular, suppose that a classical parity invariant theory is perturbatively quantized using a chiral renormalization scheme. The knowledge of a complete classification of finite renormalizations is then required to decide whether there exists a different renormalization scheme that gives a parity invariant quantization.

Another advantage of our proof is the clear separation between the applications of the locality, covariance and scaling hypotheses. We make a particular distinction between *physical scalings* (those resulting from a rescaling of the metric) and *coordinate scalings* (those resulting from the local action of some diffeomorphisms). We believe that structuring the proof in this way makes it significantly easier to generalize the result to other types of tensor or spinor fields, a task that is yet to be seriously taken up in the literature on locally covariant quantum field theory. In particular, it is likely that the crucial step in limiting the finite renormalization freedom to a quasi-finite-dimensional space is to carefully balance the covariance and scaling properties, such that there exists a coordinate system on the jets of background fields, like the *rescaled curvature coordinates* that we identified in Section 2.2, where all coordinates corresponding higher derivatives have positive weight under a combination of the physical and coordinate scalings.

Another direction in which our main result could be generalized is to consider Wick polynomials that included derivatives of fields. Our proof should extend without problems. The main difference would be that the finite renormalization coefficients C_k could then be tensor- instead of scalar-valued, since Wick polynomials with derivatives could themselves be tensor fields. This difference would affect the part of our proof where we make use of $GL(n)$-equivariance to fix the form of the A_j coefficients, which could be mixed densitized tensors. Fortunately, the main technical result on the classification of equivariant tensor densities, as stated in Lemma 2.8, is sufficiently general to apply to that case as well, since introducing densitization erases the distinction between covariant and contravariant indices.

Finally, let us say something about time-ordered products. Hollands and

Wald also gave a sketch of the proof of the characterization of finite renormalizations of time-ordered products [11, Thm.5.2], under the same hypothesis as their result about Wick polynomials. As they point out, the main difference with the case of Wick powers is in the structure of coefficients that are analogous to the C_k, which become distributions on multiple copies of the spacetime manifold. The arguments, which we encapsulated in Lemma 3.3, applying microlocal arguments to restrict the wavefront set of these distributions would have to be generalized accordingly. After that point, the proof of Theorem 3.2, would apply without essential modifications. Thus, our methods should generalize to time-ordered products as well.

The investigation of more general types of fields, of Wick polynomials with derivatives and of time-ordered products could be explored in future work.

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A Peetre's theorem with parameters

The non-linear Peetre theorem stated in Proposition 2.2 may be made significantly stronger. Let us now introduce the language needed to state the stronger version in a precise form. In the following, $\sigma\colon E \to N$ and $\rho\colon F \to M$, are two smooth bundles, where we have explicitly written the canonical projections, and we consider a map $D\colon \Gamma(E \to N) \to \Gamma(F \to M)$ between smooth sections of these bundles. We intend here to give a precise mathematical meaning to the statement that D *is local*. Before defining the most general version of locality (cf. [14, §18.16]), we consider several motivating cases of increasing complexity.

Case $N = M$. We say that D is *local* when the value $\phi(x)$, for $\phi = D[\psi] \in \Gamma(F \to M)$, depends only on the germ of $\psi \in \Gamma(E \to M)$ at $x \in M$. This version of locality is already sufficient for Propositions 2.1 and 2.2. We can loosen this notion of locality in several ways.

Case $N \neq M$. We may agree that $\phi(x)$, for $\phi = D[\psi] \in \Gamma(F \to M)$ and $x \in M$, may depend only on the germ of $\psi \in \Gamma(E \to N)$ at $y \in N$, with some fixed relationship $y = \chi(x)$, where $\chi\colon M \to N$ is some diffeomorphism. We then say that D is *χ-local*.

Case $N \neq M$ and D depends on external parameters. We can introduce a bundle $\pi\colon P \to M$, where the manifold P is interpreted as "M with parameters." Then, allowing D to depend on parameters means that D really maps sections of $E \to N$ to sections of the pullback bundle $\pi^*F \to P$, interpreted as "F with parameters." Pre-composing a section of $\pi^*F \to P$ with a section of $P \to M$ then yields a section of $F \to M$ given by a particular choice of parameters. Denoting $\eta = \chi \circ \pi$, we call the map $D\colon \Gamma(E \to N) \to \Gamma(\pi^*F \to P)$ *η-local* when $\phi(x,p) = D[\psi](x,p)$, with $(x,p) \in P$ and $\pi(x,p) = x \in M$, depends only on the germ of ψ at $y = \eta(x,p) = \chi(x) \in N$. Note that the total space of the bundle "F with parameters" can be expressed as the fibered product $\pi^*F \cong F\,{}_\rho\!\times_\pi P$ over M (where we have explicitly named the $\rho\colon F \to M$ bundle projection), which completes the pullback diagram

$$\begin{array}{ccc} F{}_\rho{\times}_\pi P & \longrightarrow & P \\ \downarrow & & \downarrow{\scriptstyle \pi} \\ F & \xrightarrow{\rho} & M \end{array} \quad . \tag{62}$$

We can illustrate all of the above maps in the diagram

$$\begin{array}{ccccc} E & & F{}_\rho{\times}_\pi P & \longrightarrow & F \\ {\scriptstyle \psi}\uparrow\downarrow & {\scriptstyle \phi = D[\psi]} & \uparrow\downarrow & {\scriptstyle \tau} & \downarrow{\scriptstyle \rho}\uparrow{\scriptstyle \phi\circ\tau} \\ N & \xleftarrow[\eta]{} & P & \xrightarrow[\pi]{} & M \\ & & \underset{\chi}{\longleftarrow} & & \end{array} \quad , \tag{63}$$

where all the solid arrows commute, the dotted arrows denote bundle sections, with $\tau \colon M \to P$ denoting a particular "choice of parameters," and $\phi \circ \tau$ was silently composed with the projection $F{}_\rho{\times}_\pi P \to F$.

General case. Finally, it is possible to relax the requirement that the map $\eta \colon P \to N$ factors as illustrated in diagram (63). The dimension of P could exceed that of N and η need not be a surjection, not even a submersion. Omitting the structure of the right square of diagram (63), we also replace $F{}_\rho{\times}_\pi P$ by a simple bundle $F \to P$, without requiring it to have the structure of a fibered product. So, given bundles $E \to N$ and $F \to P$, together with a smooth map $\eta \colon P \to N$, a map $D \colon \Gamma(E \to N) \to \Gamma(F \to P)$ is called *η-local* if $\phi(x) = D[\psi](x)$, $x \in P$, depends only on the germ of ψ at $y = \eta(x) \in N$. We can illustrate this situation by the diagram

$$\begin{array}{ccc} E & & F \\ {\scriptstyle \psi}\uparrow\downarrow & {\scriptstyle \phi = D[\psi]} & \uparrow\downarrow \\ N & \xleftarrow[\eta]{} & P \end{array} \quad , \tag{64}$$

which should be thought of as exactly analogous to diagram (63), but with the right square missing. This the rather weak notion of η-locality, with a small additional hypothesis (η non-locally constant), is actually sufficient for a non-linear version of Peetre's theorem.

Proposition A.1 (Non-linear Peetre's Theorem [14, §19.10])**.** *Let $F \to P$, $E \to N$ be smooth bundles and $\eta \colon P \to N$ a non-locally constant*[6] *smooth map, with the interpretation as in diagram (64). For every compact $K \subseteq P$ and $\psi \in \Gamma(E \to N)$, there exists an integer r, an open neighborhood $U \subseteq J^r(E \to N)$ of $j^r\psi(N) \subset U$, with $U_K \subseteq U$ the subset projecting onto $\eta(K)$, and a function $d \colon U_K \to F$ that commutes with all the projections, as illustrated by the diagram*

$$\begin{array}{ccc} J^r(E \to N) \supseteq U_K & \xrightarrow{d} & F|_K \\ \downarrow & & \downarrow \\ N \supseteq \eta(K) & \xleftarrow[\eta]{} & K \subseteq P \end{array} \quad , \tag{65}$$

[6]By *non-locally constant* we mean that for every open $U \subseteq P$ the image $\eta(U)$ contains at least two points.

such that $D[\xi](x) = d \circ j^r \xi(x)$ for any $\xi \in \Gamma(E \to N)$ with $j^r\xi(N) \subset U$. In other words, D is a differential operator of locally finite order, where locality is with respect to compact subsets of P and compact open neighborhoods in $\Gamma(E \to N)$.

Sketch of proof. With the definitions as discussed above, the proposition is essentially a restatement of Theorem 19.10 of [14], which follows directly from Theorem 19.7 and Corollary 19.8 that precede it. We refer the reader to the book [14] for full details. Let us simply mention that, in general outline, the proof proceeds by contradiction. If D depended non-trivially on an infinite number of derivatives of its argument, then it would be possible to engineer a smooth section ψ such that $D[\psi]$ could not itself be smooth. □

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