

Wedge-Local Quantum Fields on a Nonconstant Noncommutative Spacetime

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Abstract

Within the framework of warped convolutions we deform the massless free scalar field. The deformation is performed by using the generators of the special conformal transformations. The investigation shows that the deformed field turns out to be wedge-local. Furthermore, it is shown that the spacetime induced by the deformation with the special conformal operators is nonconstant noncommutative. The noncommutativity is obtained by calculating the deformed commutator of the coordinates.

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1 Introduction

Noncommutative quantum field theories (NCQFT) enjoy wide popularity among theoretical physicists. From a string theoretical point of view, NCQFT became popular due to the observation that it can be obtained in a certain low energy limit from string theory [24]. From a quantum field theoretical aspect, NCQFT gained interest due to many reasons. Most importantly it was thought that by the introduction of a fundamental length renormalisation ambiguities would disappear and ultra-violet (UV) divergences would be canceled. But already in first order of perturbation theory, the euclidean noncommutative ϕ^4 model exhibited a new type of divergences. The new divergences could not be cured by standard renormalisation procedures. In a series of papers [14, 15, 16] the authors added a term to the noncommutative ϕ^4 model, based on duality considerations, and proved the renormalisability to all orders in perturbation.

Quantum field theory on a noncommutative Minkowski spacetime was rigorously realised in [8]. The quantum field therein was defined on a tensor product space $\mathcal{V} \otimes \mathcal{H}$. Where \mathcal{H} is the Bosonic Fock space and \mathcal{V} is the representation space of the noncommuting coordinate operators \hat{x}_ν , satisfying the Moyal-Weyl plane commutator relations $[\hat{x}_\mu, \hat{x}_\nu] = i\theta_{\mu\nu}$. The matrix $\theta_{\mu\nu}$ is a constant, nondegenerate and skew-symmetric matrix. Many authors [3, 11, 12] succeeded representing the scalar field on \mathcal{H} instead of $\mathcal{V} \otimes \mathcal{H}$. Furthermore, in [12] this representation was used to construct a map from the set of skew-symmetric matrices, which describe the noncommutativity, to a set of wedges. In the next step the construction was applied to map the noncommutative scalar field to a scalar field living on a wedge. The respective model led to weakened locality and covariance properties of the field and to a

nontrivial S-matrix. The result is astonishing because notions of covariance and locality are usually lost on a noncommutative spacetime.

The method of deformation was further generalised in [6, 7, 17] and was made public under the name of warped convolutions. It is interesting to note that the model formulated in [12] can be obtained from warped convolutions by using the momentum operator P_μ for the deformation. The method in [17] was also successfully used to define deformations of a scalar massive Fermion, [2].

In fact any strongly continuous unitary representation of the group \mathbb{R}^n can be used to deform the free scalar field. In the following work we deform the QFT of a free scalar field with the special conformal operators using the method of warped convolutions. We show that the resulting noncommutative spacetime is nonconstant. Furthermore, we show that the constructed model exhibits covariance and locality properties which are highly nontrivial for a nonconstant noncommutative spacetime.

The organisation of the paper is as follows: In Sec. 2, we give a brief introduction of the conformal group and the isomorphism to the pseudo-orthogonal group $SO(2, d)$. The proof of self-adjointness of the special conformal operators was given rigorously in [29] and is sketched in Sec. 3. The proof therein relies on the fact that the momentum operator and the special conformal operator are unitarily equivalent. Furthermore, we deform the free scalar field with the special conformal operators and use the unitary equivalence to prove convergence of the deformation in the Hilbert space norm. The Wightman properties, transformation properties and wedge-locality of the deformed field are proven in Sec. 4. In Sec. 5, we show how the deformation with the special conformal operators leads to a nonconstant noncommutative spacetime. This is done by calculating the commutator of the coordinates using the deformed product given in [7].

2 The conformal group and $SO(2, d)$

2.1 Generators of the conformal group

A conformal transformation of the coordinates is defined to be an invertible mapping $\mathbf{x}' \rightarrow \mathbf{x}$, which leaves the d -dimensional metric g invariant up to a scale factor, [9]:

$$g'_{\mu\nu}(x') = F(x)g_{\mu\nu}(x). \quad (1)$$

The mappings satisfying the condition (1) are the Lorentz transformations, the translations, the dilations and the special conformal transformations. These transformations are generated by the operators $L_{\mu\nu}$, P_ρ , D , K_σ and the set of all conformal transformations forms the conformal group.

The conformal algebra is defined by the commutation relations of the generators and is given as follows:

$$[L_{\mu\nu}, L_{\rho\sigma}] = i(\eta_{\mu\sigma}L_{\nu\rho} + \eta_{\nu\rho}L_{\mu\sigma} - \eta_{\mu\rho}L_{\nu\sigma} - \eta_{\nu\sigma}L_{\mu\rho}), \quad (2)$$

$$[P_\rho, L_{\mu\nu}] = i(\eta_{\rho\mu}P_\nu - \eta_{\rho\nu}P_\mu), \quad [K_\rho, L_{\mu\nu}] = i(\eta_{\rho\mu}K_\nu - \eta_{\rho\nu}K_\mu), \quad (3)$$

$$[P_\rho, D] = iP_\rho, \quad [K_\rho, D] = -iK_\rho, \quad (4)$$

$$[P_\rho, K_\mu] = 2i(\eta_{\rho\mu}D - L_{\rho\mu}), \quad (5)$$

with all other commutators being equal to 0.

2.2 Isomorphism between the conformal group and $SO(2, d)$

To see the isomorphism between the conformal group in d dimensions and the pseudo-orthogonal group $SO(2, d)$, one introduces the following definitions:

$$J_{4,\mu} := \frac{1}{2}(P_\mu - K_\mu), \quad J_{5,\mu} := \frac{1}{2}(P_\mu + K_\mu), \quad (6)$$

$$J_\mu^\pm := J_{5,\mu} \pm J_{4,\mu} \quad J_{-1,0} := D, \quad J_{\mu\nu} := L_{\mu\nu}, \quad (7)$$

$$J_{ab} = -J_{ba}, \quad a, b = 0, 1, \dots, d, d+1. \quad (8)$$

The defined generators J_{ab} obey the algebra of $SO(2, d)$ with the following commutator relations:

$$[J_{ab}, J_{cd}] = i(\eta_{ad}J_{bc} + \eta_{bc}J_{ad} - \eta_{ac}J_{bd} - \eta_{bd}J_{ac}), \quad (9)$$

where the diagonal metric has the following form

$$\eta_{aa} = (+1, \underbrace{-1, \dots, -1}_d, +1). \quad (10)$$

This shows the isomorphism between the conformal group and $SO(2, d)$. As one can easily see, the full conformal group contains the Poincaré group as a subgroup.

3 Deforming the scalar quantum field

The method used for deformation in this work was introduced in [6, 7] and goes by the name of warped convolutions. This device can be used to deform relativistic quantum field theories and the hope is in near future to solve a nontrivial interacting QF model in 4 dimensions. There were interesting results in [13] where the authors obtained a non trivial S-matrix differing from the free one by momentum dependent phases. The results introduced in [12, 13] can be obtained by using the framework of warped convolutions to deform the underlying QFT. This is done by using the momentum operator as the generator of a strongly continuous unitary representation of the group \mathbb{R}^n .

Our approach in this work will be to use the special conformal operators as the generators of such representations. This will lead to a new QFT model which on one hand can be interpreted as a QF on a nonconstant noncommutative spacetime, and on the other hand as a wedge-local QFT model.

3.1 Self-adjointness of the special conformal operators

To proceed with deforming via warped convolutions, it is necessary to prove self-adjointness of the special conformal operator K_μ . The proof was given in [29] relying on the fact that the special conformal operator can be defined as

$$K_\mu := U_R P_\mu U_R, \quad (11)$$

where U_R is the inversion operator. The reason for the definition is that any special conformal transformation

$$U(b)x^\mu = \frac{x_\mu - b_\mu x^2}{1 - 2b_\mu x^\mu + b^2 x^2}, \quad (12)$$

can be written as a product

$$U(b) = U_R T(b) U_R, \quad (13)$$

of a translation $T(b)x^\mu = x^\mu + b^\mu$ and inversions $U_R x^\mu = -x^\mu/x^2$. By constructing a self-adjoint unitary representation U_R in $\mathcal{H}_1 := L^2(d^n \mu(\mathbf{p}), \mathbb{R}^n) = \{f : \int d^n \mu(\mathbf{p}) |f(\mathbf{p})|^2 < \infty, d^n \mu(\mathbf{p}) := d^n \mathbf{p} (2|\mathbf{p}|)^{-1}\}$ for $n \geq 1$, the essential self-adjointness of the operators K^μ on the dense domain $\Delta(R) := U_R \Delta(P)$ follows. $\Delta(P)$ is the dense domain of all functions from \mathcal{H}_1 vanishing at infinity faster than any inverse polynomial in p^k and is given as follows

$$\Delta(P) = \{f \in \mathcal{H}_1 : |(\mathbf{p}^2)^r f(\mathbf{p})| \leq c_r(f) < \infty; \quad r = 0, 1, 2, \dots\}. \quad (14)$$

Due to the unitary equivalence (11), P_μ and K_μ have the same spectrum contained in the closed forward cone

$$\overline{V_0^+} := \{p^\mu : p^\mu p_\mu \geq 0, p_0 \geq 0\}. \quad (15)$$

The last step in [29] consists in showing that the special conformal operator defined in the following way (11), is identical with the special conformal generator of the conformal group.

3.2 Special conformal transformation of the free scalar field

Since in the context of the present paper we need the transformation of the free scalar field under the special conformal group, we shall also briefly summarise those results obtained in [29].

For $n = 1$ the existence of a unitary representation for the whole conformal group was proven. The special conformal operator transforms the free scalar field $\phi(x)$ in the following manner

$$\alpha_b(\phi(x)) := e^{ib_\mu K^\mu} \phi(x) e^{-ib_\mu K^\mu} = \phi(x_b) - \phi\left(-\frac{b}{b^\mu b_\mu}\right), \quad (16)$$

where

$$x_b^\mu := \frac{x_\mu - b_\mu x^2}{1 - 2b_\mu x^\mu + b^2 x^2}. \quad (17)$$

In the two dimensional spacetime test functions $f \in \mathcal{S}(\mathbb{R}^2)$ which are used to smear the distribution valued operator $\phi(x)$ are chosen to satisfy $\int d^2 x f(x) = 0$. The reason for this specific choice is to circumvent IR-divergences and it will be used through the entire work.

Now if $n = 2l + 1$ for $l \in \mathbb{N}$ one obtains the following result

$$\alpha_b(\phi(x)) = \sigma_b(x)^{\frac{1-n}{2}} \phi(x_b), \quad (18)$$

where

$$\sigma_b(x) := 1 - 2b_\mu x^\mu + b^2 x^2. \quad (19)$$

It was further proven that one only obtains a unitary representation for the whole conformal group if $n = 4l + 1$ for $l \in \mathbb{N}$. In the other cases for odd n one has to deal with representations of the covering of the conformal group. The reason for the non-existence of a unitary representation for the whole conformal group lies in the non positivity of the scale factor $\sigma_b(x)$. For the present paper this will become important due to our intention to formulate the model in four spacetime dimensions. In Sec. 4, we will prove that the scale factor $\sigma_b(x)$ is positive for a scalar field localised in the wedge. Therefore, we will not have a problem obtaining a unitary representation for the whole conformal group.

3.3 Deforming the QF with special conformal operators

In this section we deform the massless scalar field with the special conformal operators using the framework warped convolutions. To proceed with the deformation, we first define the undeformed free scalar field ϕ with mass $m = 0$ on the $n+1$ -dimensional Minkowski spacetime as an operator valued distribution acting on its domain in the Bosonic Fock space. Such a particle with momentum $\mathbf{p} \in \mathbb{R}^n$ has the energy defined by $\omega_{\mathbf{p}} = |\mathbf{p}|$.

Definition 3.1. *The Bosonic Fock space \mathcal{H}^+ is defined as in [10, 25]:*

$$\mathcal{H}^+ = \bigoplus_{m=0}^{\infty} \mathcal{H}_m^+$$

where the m particle subspaces are given as

$$\begin{aligned} \mathcal{H}_m^+ &= \{ \Psi_m : \partial V_+ \times \cdots \times \partial V_+ \rightarrow \mathbb{C} \text{ symmetric} | \\ &\| \Psi_m \|^2 = \int d^n \mu(\mathbf{p}_1) \cdots \int d^n \mu(\mathbf{p}_m) |\Psi_m(\mathbf{p}_1, \dots, \mathbf{p}_m)|^2 < \infty \}, \end{aligned}$$

with

$$\partial V_+ := \{ p \in \mathbb{R}^d | p^2 = 0, p_0 > 0 \}.$$

The particle annihilation and creation operators can be defined by their action on m -particle wave functions

$$\begin{aligned} (a(f)\Psi)_m(\mathbf{p}_1, \dots, \mathbf{p}_m) &= \sqrt{m+1} \int d^n \mu(\mathbf{p}) \overline{f(\mathbf{p})} \Psi_{m+1}(\mathbf{p}, \mathbf{p}_1, \dots, \mathbf{p}_m) \\ (a(f)^*\Psi)_m(\mathbf{p}, \mathbf{p}_1, \dots, \mathbf{p}_m) &= \begin{cases} 0, & m = 0 \\ \frac{1}{\sqrt{m}} \sum_{k=1}^m f(\mathbf{p}_k) \Psi_{m-1}(\mathbf{p}_1, \dots, \mathbf{p}_{k-1}, \mathbf{p}_{k+1}, \dots, \mathbf{p}_m), & m > 0 \end{cases} \end{aligned}$$

with $f \in \mathcal{H}_1$ and $\Psi_m \in \mathcal{H}_m^+$. The commutator relations of $a(f), a(f)^*$ follow immediately and are given as follows

$$[a(f), a(g)^*] = (f, g) = \int d^n \mu(\mathbf{p}) \overline{f(\mathbf{p})} g(\mathbf{p}), \quad [a(f), a(g)] = 0 = [a^*(f), a^*(g)].$$

Particle annihilation and creation operators with sharp momentum are introduced as operator valued distributions and are given as follows

$$a(f) = \int d^n \mu(\mathbf{p}) \overline{f(\mathbf{p})} a(\mathbf{p}), \quad a(f)^* = \int d^n \mu(\mathbf{p}) f(\mathbf{p}) a^*(\mathbf{p}),$$

where the particle annihilation and creation operators with sharp momentum satisfy the following commutator relations

$$[a(\mathbf{p}), a(\mathbf{q})^*] = 2\omega_{\mathbf{p}}\delta^n(\mathbf{p} - \mathbf{q}), \quad [a(\mathbf{p}), a(\mathbf{q})] = 0 = [a^*(\mathbf{p}), a^*(\mathbf{q})].$$

In the next step we define the warped convolutions of the free scalar field. This is done using the essential self-adjointness of the generators K_μ which in turn define a unitary operator $U(b) := e^{ib_\mu K^\mu}$. The definition of the operator valued function $U(b)$ leads to a strongly continuous unitary representation of \mathbb{R}^d , for each $b_\mu \in \mathbb{R}^d$. This can be proven by making use of Stone's theorem, [22]. To define the deformation, we need the the unitary operator of translations defined by $T(y) := e^{iy_\mu P^\mu}$, for each $y_\mu \in \mathbb{R}^d$, and the extended dense domain $\Delta_m(P) := \bigotimes_{k=1}^m \Delta(P)$. Furthermore, we define $\Gamma(U_R) := \bigotimes_{k=1}^m U_R$ to be the unitary operator of the inversions on \mathcal{H}_m^+ , [22]. From the former definitions the extended domain $\Delta_m(R) = \Gamma(U_R)\Delta_m(P)$ follows.

Definition 3.2. *Let θ be a real skew-symmetric matrix w.r.t. the Minkowski scalar-product on \mathbb{R}^d and let $\phi(f)$ be the free scalar field smeared out with functions $f \in \mathcal{S}(\mathbb{R}^d)$. Then the operator valued distribution $\phi(f)$ deformed with the special conformal operators, denoted as $\phi_{\theta,K}(f)$, is defined on vectors of the dense domain $\Delta_m(R)$ as follows*

$$\phi_{\theta,K}(f)\Psi_m := (2\pi)^{-d} \iint d^d y d^d k e^{-iy_\mu k^\mu} \alpha_{\theta y}(\phi(f)) U(k) \Psi_m \quad (20)$$

$$= (2\pi)^{-d} \iint d^d y d^d k e^{-iy_\mu k^\mu} \alpha_{\theta y} \left(a(\overline{f^-}) + a^*(f^+) \right) U(k) \Psi_m \quad (21)$$

$$=: \left(a_{\theta,K}(\overline{f^-}) + a_{\theta,K}^*(f^+) \right) \Psi_m. \quad (22)$$

The test functions $f^\pm(\mathbf{p})$ in momentum space are defined as follows

$$f^\pm(\mathbf{p}) := \int d^d x f(x) e^{\pm i p x}, \quad p = (\omega_{\mathbf{p}}, \mathbf{p}) \in \partial V_+.$$

By arguing with the essential self-adjointness of the special conformal operator, it can be shown that the integral (20) converges in the strong operator topology if the undeformed operator is bounded, [7]. Due to the fact that we are dealing with an unbounded operator it is not clear in what sense the deformation with the special conformal operator converges. To show that the warped convolutions (20) exist in Hilbert space norm, we use the unitary equivalence (11) between the momentum operator and the special conformal operator.

Lemma 3.3. *For $f \in \mathcal{S}(\mathbb{R}^d)$ and $\Psi_m \in \Delta_m(R)$, a transformation exists that maps the field deformed with the momentum operator $\phi_{\theta,P}(f)$ to the field deformed with the special conformal operator $\phi_{\theta,K}(f)$. This transformation is given as follows*

$$\phi_{\theta,K}(f)\Psi_m = \Gamma(U_R) (\Gamma(U_R)\phi(f)\Gamma(U_R))_{\theta,P} \Gamma(U_R)\Psi_m.$$

Proof. By using the unitary equivalence (11) the lemma is easily proven

$$\begin{aligned} \phi_{\theta,K}(f)\Psi_m &= (2\pi)^{-d} \iint d^d y d^d k e^{-iy_\mu k^\mu} U(\theta y) \phi(f) U(-\theta y + k) \Psi_m \\ &= (2\pi)^{-d} \iint d^d y d^d k e^{-iy_\mu k^\mu} \Gamma(U_R) T(\theta y) \Gamma(U_R) \phi(f) \Gamma(U_R) T(-\theta y + k) \Gamma(U_R) \Psi_m \\ &= \Gamma(U_R) (\Gamma(U_R)\phi(f)\Gamma(U_R))_{\theta,P} \Gamma(U_R)\Psi_m. \end{aligned}$$

□

Lemma 3.4. For $\Phi_m \in \Delta_m(R) \subset \mathcal{H}_m^+$ the familiar bounds of the free field hold for the deformed field $\phi_{\theta,K}(f)$ and therefore the deformation with the special conformal operators exists in the Hilbert space norm.

Proof. By using lemma 3.3 one obtains the familiar bounds for a free scalar field. For $\Phi_m \in \Delta_m(R)$ there exists a $\Psi_m \in \Delta_m(P)$ such that the following holds

$$\begin{aligned} \|\phi_{\theta,K}(f)\Phi_m\| &= \|\phi_{\theta,K}(f)\Gamma(U_R)\Psi_m\| = \|(\Gamma(U_R)\phi(f)\Gamma(U_R))_{\theta,P}\Psi_m\| = \|(\phi(U_R f))_{\theta,P}\Psi_m\| \\ &\leq \left\| (a(U_R \bar{f}^-))_{\theta,P}\Psi_m \right\| + \left\| (a^*(U_R f^+))_{\theta,P}\Psi_m \right\| \leq \|U_R f^+\|^2 \left\| (N+1)^{1/2}\Psi_m \right\|^2 + \\ &\|U_R f^-\|^2 \left\| (N+1)^{1/2}\Psi_m \right\|^2 = \|f^+\|^2 \left\| (N+1)^{1/2}\Psi_m \right\|^2 + \|f^-\|^2 \left\| (N+1)^{1/2}\Psi_m \right\|^2, \end{aligned}$$

where in the last lines we used the Cauchy-Schwarz inequality, the bounds given in [12] and the unitarity of U_R . \square

4 Properties of the deformed quantum field

In the following section we prove the Wightman properties of the deformed field. The Wightman axioms of covariance and locality are not satisfied, but are replaced by wedge covariance and wedge-locality. The relation between the fields defined on a deformed space-time and fields defined on the wedge is given by the the constructed map in [7, 12]. To use this map we give the transformation property of the deformed quantum field ϕ_θ under Lorentz transformations and thus relate the skew-symmetric matrices to wedges. Furthermore, we prove that the field obtained by the construction is a wedge Lorentz-covariant and wedge-local quantum field.

4.1 Wightman properties of the deformed QF

In this section we prove that the deformed field $\phi_{\theta,K}$ satisfies the Wightman properties with the exception of covariance and locality.

Proposition 4.1. Let θ be a real skew-symmetric matrix w.r.t. the Minkowski scalar-product on \mathbb{R}^d and $f \in \mathcal{S}(\mathbb{R}^d)$.

a) The dense subspace \mathcal{D} of vectors of finite particle number is contained in the domain $\mathcal{D}^{\theta,K} = \{\Psi \in \mathcal{H} \mid \|\phi_{\theta,K}(f)\Psi\|^2 < \infty\}$ of any $\phi_{\theta,K}(f)$. Moreover, $\phi_{\theta,K}(f)\mathcal{D} \subset \mathcal{D}$ and $\phi_{\theta,K}(f)\Omega = \phi(f)\Omega$.

b) For scalar fields deformed via warped convolutions and $\Psi \in \mathcal{D}$,

$$f \longmapsto \phi_\theta(f)\Psi \tag{23}$$

is a vector valued tempered distribution.

c) For $\Psi \in \mathcal{D}$ and $\phi_{\theta,K}(f)$ the following holds

$$\phi_{\theta,K}(f)^*\Psi = \phi_{\theta,K}(\bar{f})\Psi. \tag{24}$$

For real $f \in \mathcal{S}(\mathbb{R}^d)$, the deformed field $\phi_\theta(f)$ is essentially self-adjoint on \mathcal{D} .

d) The Reeh-Schlieder property holds: Given an open set of spacetime $\mathcal{O} \subset \mathbb{R}^d$ then

$$\mathcal{D}_\theta(\mathcal{O}) := \text{span}\{\phi_\theta(f_1) \dots \phi_\theta(f_m)\Omega : m \in \mathbb{N}, f_1 \dots f_m \in \mathcal{S}(\mathcal{O})\} \tag{25}$$

is dense in \mathcal{H} .

Proof. a) The fact that $\mathcal{D} \subset \mathcal{D}^\theta$, follows directly from lemma 3.4 because the deformed scalar field satisfies the well known bounds of the free field. The fact that the deformed field acting on the vacuum is the same as the free field acting on Ω , can be easily shown due to the property of the unitary operators $U(b)\Omega = \Omega$.

b) By using lemma 3.4 one can see that the right hand side depends continuously on the function f , hence the temperateness of $f \mapsto \phi_{\theta,K}(f)\Psi$, $\Psi \in \mathcal{D}$ follows.

c) The hermiticity of the deformed field $\phi_{\theta,K}(f)^*$ is proven in the following

$$\begin{aligned} \phi_{\theta,K}(f)^*\Psi &= (2\pi)^{-d} \left(\iint d^d y d^d k e^{-iy_\mu k^\mu} \alpha_{\theta y}(\phi(f)) U(k) \right)^* \Psi \\ &= (2\pi)^{-d} \iint d^d y d^d k e^{iy_\mu k^\mu} U(-k) \alpha_{\theta y}(\phi(f))^* \Psi \\ &= (2\pi)^{-d} \iint d^d y d^d k e^{iy_\mu k^\mu} \alpha_{\theta y}(\phi(\bar{f})) U(-k) \Psi = \phi_{\theta,K}(\bar{f})\Psi, \quad \Psi \in \mathcal{D}. \end{aligned}$$

In the last lines we performed a variable substitution ($y \rightarrow y + \theta^{-1}k$) and ($k \rightarrow -k$). The essential self-adjointness of the deformed field for real f can be shown along the same lines as in [5].

d) For the proof of the Reeh-Schlieder property we will make use of the unitary equivalence (11). First note that the spectral properties of the representation of the special conformal transformations $U(y)$ are the same as for the representation of translations. This leads to the application of the standard Reeh-Schlieder argument [28] which states that that $\mathcal{D}_\theta(\mathcal{O})$ is dense in \mathcal{H} if and only if $\mathcal{D}_\theta(\mathbb{R}^d)$ is dense in \mathcal{H} . We choose the functions $f_1, \dots, f_m \in \mathcal{S}(\mathbb{R}^d)$ such that the Fourier transforms of the functions do not intersect the past light cone and therefore the domain $\mathcal{D}_\theta(\mathbb{R}^d)$ consists of the following vectors

$$\begin{aligned} \Gamma(U_R)\phi_{\theta,K}(f_1) \dots \phi_{\theta,K}(f_m)\Omega &= \Gamma(U_R)a_{\theta,K}^*(f_1^+) \dots a_{\theta,K}^*(f_m^+)\Omega \\ &= \Gamma(U_R)\Gamma(U_R)(\Gamma(U_R)a^*(f_1^+)\Gamma(U_R))_{\theta,P} \dots (\Gamma(U_R)a^*(f_m^+)\Gamma(U_R))_{\theta,P}\Gamma(U_R)\Omega \\ &= a_{\theta,P}^*(U_R f_1^+) \dots a_{\theta,P}^*(U_R f_m^+)\Omega = \sqrt{m!}P_m(S_m(U_R f_1^+ \otimes \dots \otimes U_R f_m^+)), \end{aligned}$$

where P_m denotes the orthogonal projection from $\mathcal{H}_1^{\otimes m}$ onto its totally symmetric subspace \mathcal{H}_m^+ , and $S_m \in \mathcal{B}(\mathcal{H}_1^{\otimes m})$ is the multiplication operator given as

$$S_m(p_1, \dots, p_m) = \prod_{1 \leq l < k \leq m} e^{ip_l \theta p_k}.$$

Since the operator $\Gamma(U_R)$ is a unitary operator the functions $U_R f_k^+$ for $f_k^+ \in \mathcal{S}(\mathbb{R}^d)$ will give rise to dense sets of functions in \mathcal{H}_1 . Following the same arguments as in [12] the density of $\mathcal{D}_\theta(\mathbb{R}^d)$ in \mathcal{H} follows. Note that we proved the density for vectors $\Gamma(U_R)\phi_{\theta,K}(f_1) \dots \phi_{\theta,K}(f_m)\Omega$ and not for the vectors without $\Gamma(U_R)$ as stated in the proposition. We use the unitarity of $\Gamma(U_R)$ to argue that vectors dense in \mathcal{H} stay dense after the application of a unitary operator. \square

4.2 Wedge-covariant fields

The authors in [12] constructed a map $Q : W \mapsto Q(W)$ from a set $\mathcal{W}_0 := \mathcal{L}_+^\uparrow W_1$ of wedges, where $W_1 := \{x \in \mathbb{R}^d : x_1 > |x_0|\}$ to a set $\mathcal{Q}_0 \subset \mathbb{R}_{d \times d}^-$ of skew-symmetric matrices. In the

next step they considered the corresponding fields $\phi_W(x) := \phi(Q(W), x)$. The meaning of the correspondence is that the field $\phi(Q(W), x)$ is a scalar field living on a NC spacetime which can be equivalently realised as a field defined on the wedge.

To show the covariance properties of the deformed quantum fields we use the homomorphism $Q(W)$ to map the deformed scalar fields to quantum fields defined on a wedge. Let us first define the following map.

Definition 4.2. *Let θ be a real skew-symmetric matrix on \mathbb{R}^d then the map $\gamma_\Lambda(\theta)$ is defined as follows*

$$\gamma_\Lambda(\theta) := \begin{cases} \Lambda\theta\Lambda^T, & \Lambda \in \mathcal{L}^\uparrow, \\ -\Lambda\theta\Lambda^T, & \Lambda \in \mathcal{L}^\downarrow. \end{cases} \quad (26)$$

Definition 4.3. *θ is called an admissible matrix if the realisation of the homomorphism $Q(\Lambda W)$ defined by the map $\gamma_\Lambda(\theta)$ is a well defined mapping. This is the case iff θ has in n dimensions the following form*

$$\begin{pmatrix} 0 & \lambda & 0 & \cdots & 0 \\ \lambda & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, \quad \lambda \geq 0. \quad (27)$$

For the physical most interesting case of 4 dimensions the skew-symmetric matrix θ has the more general form

$$\begin{pmatrix} 0 & \lambda & 0 & 0 \\ \lambda & 0 & 0 & 0 \\ 0 & 0 & 0 & \eta \\ 0 & 0 & -\eta & 0 \end{pmatrix}, \quad \lambda \geq 0, \eta \in \mathbb{R}. \quad (28)$$

Before we use the map from the set of skew-symmetric matrices to the wedges we state the following lemma about the transformation properties of the deformed field.

Lemma 4.4. *The transformation of the deformed particle annihilation and creation operator $a_{\theta,K}(\mathbf{p}), a_{\theta,K}^*(\mathbf{p})$, for $\mathbf{p} \in \partial V_+$ and θ being admissible, under the adjoint action $V(0, \Lambda)$ of the Lorentz group, $\Lambda \in \mathcal{L}$, is the following*

$$V(0, \Lambda)a_{\theta,K}(\mathbf{p})V(0, \Lambda)^{-1} = a_{\gamma_\Lambda(\theta),K}(\pm\Lambda\mathbf{p}), \quad (29)$$

$$V(0, \Lambda)a_{\theta,K}^*(\mathbf{p})V(0, \Lambda)^{-1} = a_{\gamma_\Lambda(\theta),K}^*(\pm\Lambda\mathbf{p}), \quad (30)$$

where the first sign is for $\Lambda \in \mathcal{L}^\uparrow$ and the second sign is for $\Lambda \in \mathcal{L}^\downarrow$. Hence the deformed field $\phi_{\theta,K}(x)$ transforms

$$V(0, \Lambda)\phi_{\theta,K}(x)V(0, \Lambda)^{-1} = \phi_{\gamma_\Lambda(\theta),K}(\Lambda x). \quad (31)$$

Proof. The proof is done along the lines of [7]. $V(0, \Lambda)$ is a unitary operator for $\Lambda \in \mathcal{L}^\uparrow$ and an antiunitary operator for $\Lambda \in \mathcal{L}^\downarrow$. Due to the commutator relation of the special conformal operator and the generator of the Lorentz transformations one obtains

$$V(0, \Lambda)U(x)V(0, \Lambda)^{-1} = U(\Lambda x), \quad x \in \mathbb{R}^d. \quad (32)$$

Therefore, the deformed scalar field $\phi_{\theta,K}$ transforms under the adjoint action of the Lorentz group in the following way

$$\begin{aligned} V(0, \Lambda)\phi_{\theta,K}(x)V(0, \Lambda)^{-1} &= (2\pi)^{-d}V(0, \Lambda) \iint d^d y d^d k e^{-iy_\mu k^\mu} \alpha_{\theta y}(\phi(x))U(k)V(0, \Lambda)^{-1} \\ &= (2\pi)^{-d} \iint d^d y d^d k e^{-i\sigma y_\mu k^\mu} \alpha_{\Lambda\theta y}(V(0, \Lambda)\phi(x)V(0, \Lambda)^{-1})U(\Lambda k) \\ &= (2\pi)^{-d} \iint d^d y d^d k e^{-iy_\mu k^\mu} \alpha_{\gamma_\Lambda(\theta)y}(\phi(\Lambda x))U(k) \\ &= \phi_{\gamma_\Lambda(\theta),K}(\Lambda x), \end{aligned}$$

where σ is $+1$ if V is unitary and -1 if V is antiunitary. Moreover in the last lines we performed the integration variable substitutions $(y, k) \rightarrow (\sigma\Lambda^T y, \Lambda^{-1}k)$. \square

In the next step we use the homomorphism (26) to map the deformed field to a field defined on a wedge. Furthermore we show that the field deformed with the special conformal operator is a wedge-covariant quantum field which transforms covariantly under the adjoint action of the Lorentz group $V(0, \Lambda)$. For this purpose let us first introduce the notion of a wedge-covariant quantum field, [12].

Definition 4.5. *Let $\phi = \{\phi_W : W \in \mathcal{W}_0\}$ denote the family of fields satisfying the domain and continuity assumptions of the Wightman axioms. Then the field ϕ is defined to be a wedge Lorentz-covariant quantum field if the following condition is satisfied:*

- For any $W \in \mathcal{W}_0$ and $f \in \mathcal{S}(\mathbb{R}^d)$ the following holds

$$\begin{aligned} V(\Lambda)\phi_W(f)V(\Lambda)^{-1} &= \phi_{\Lambda W}(f \circ (\Lambda)^{-1}), \quad \Lambda \in \mathcal{L}_+^\uparrow, \\ V(j)\phi_W(f)V(j)^{-1} &= \phi_{jW}(\bar{f} \circ j)^{-1}. \end{aligned}$$

We use the homomorphism $Q : W \mapsto Q(W)$ to define the deformed fields as quantum fields defined on the wedge, this is done in the following way

$$\phi_W(f) := \phi(Q(W), f) = \phi_{\theta,K}(f). \quad (33)$$

Proposition 4.6. *The family of fields $\phi = \{\phi_W : W \in \mathcal{W}_0\}$ defined by the deformation with the special conformal operators are wedge-covariant quantum fields on the Bosonic Fock space, w.r.t. the unitary representation V of the Lorentz group.*

Proof. Following lemma (4.4), the deformed field $\phi_{\theta,K}(x)$ transforms under the adjoint action V of the Lorentz group in the following way

$$V(0, \Lambda)\phi_W(x)V(0, \Lambda)^{-1} = V(0, \Lambda)\phi_{\theta,K}(x)V(0, \Lambda)^{-1} = \phi_{\gamma_\Lambda(\theta),K}(\Lambda x) = \phi_{\Lambda W,K}(\Lambda x), \quad (34)$$

where in the last lines we applied the map $Q(\Lambda W) = \gamma_\Lambda(Q(W)) = \gamma_\Lambda(\theta)$. Therefore, one obtains the wedge-covariance property of the scalar field under the Lorentz group. \square

A few comments are in order. The covariance property is given in the 4 dimensional case as well. As explained in Sec. 3, a unitary representation for the whole conformal group does not exist due to the absolute value of the scale factor. We will show in the next section that the scale factor is positive for a field localised in the wedge and therefore one has a unitary representation of the whole conformal group.

4.3 Wedge-local fields

In this section we will show that the wedge-covariant quantum field defined in the last section, is a *wedge-local* field. We first define the notion of the wedge-local field.

Definition 4.7. *The fields $\phi = \{\phi_W : W \in \mathcal{W}_0\}$ are said to be wedge-local if the following commutator relation is satisfied*

$$[\phi_{W_1}(f), \phi_{-W_1}(g)]\Psi = 0, \quad \Psi \in \mathcal{D}, \quad (35)$$

for all $f, g \in C_0^\infty(\mathbb{R}^d)$ with $\text{supp } f \subset W_1$ and $\text{supp } g \subset -W_1$.

To show that the fields defined in the last section are wedge-local, we use the following proposition (2.10, [7]) and lemma 4.9.

Proposition 4.8. *Let the scalar fields $\phi(f), \phi(g)$ be such that $[\alpha_{\theta v}(\phi(f)), \alpha_{-\theta u}(\phi(g))] = 0$ for all $v, u \in \text{sp}U$ and for $f, g \in C_0^\infty(\mathbb{R}^d)$. Then*

$$[\phi_{\theta, \kappa}(f), \phi_{-\theta, \kappa}(g)]\Psi = 0, \quad \Psi \in \mathcal{D}. \quad (36)$$

Lemma 4.9. *The special conformal transformations $U_{\theta v}$, with $v \in \text{sp}U$ and θ being admissible, map the right wedge into the right wedge $U_{\theta v}(W_1) \subset W_1$. Furthermore, the special conformal transformations $U_{-\theta u}$, with $u \in \text{sp}U$ and θ being admissible, map the left wedge into the left wedge $U_{-\theta v}(-W_1) \subset -W_1$.*

Proof. We first prove for $x^\mu \in W_1$, $v \in \text{sp}U$, θ being admissible and $\kappa > 0$, that the vector $x'^\mu := x^\mu + \kappa(\theta v)^\mu \in W_1$.

$$\begin{aligned} x'^1 &> |x'^0| \\ x^1 + \kappa\lambda v^0 &> |x^0 + \kappa\lambda v^1|. \end{aligned}$$

The right hand side is obviously greater than zero and therefore we square both sides and obtain

$$\kappa^2\lambda^2(v_0^2 - v_1^2) - (x_0^2 - x_1^2) - 2\kappa\lambda(v^1x^0 - v^0x^1) > 0.$$

Due to the fact that the sum of the first two terms is greater than zero, we are only left with proving that the following inequality

$$\lambda v^1 x^0 - \lambda v^0 x^1 \leq 0$$

is satisfied. Equality only holds if $v_0 = 0$ or λ is zero. So if $v_0, \lambda \neq 0$ we have to show the following

$$x^1 > \frac{v^1}{v^0} x^0. \quad (37)$$

(37) is satisfied, because the stronger inequality

$$x_1 > \underbrace{\left| \frac{v_1}{v_0} \right|}_{0 < \dots < 1} |x_0|$$

holds. By using the vector x'^μ we now can easily prove that $x_{\theta v}^\mu \in W_1$. To show that $x_{\theta v}^\mu \in W_1$ the following inequality must be satisfied.

$$x_{\theta v}^1 > |x_{\theta v}^0| \quad (38)$$

$$(x^1 - (\theta v)^1 x^2)/(1 - 2(\theta v) \cdot x + (\theta v)^2 x^2) > |(x^0 - (\theta v)^0 x^2)/(1 - 2(\theta v) \cdot x + (\theta v)^2 x^2)|. \quad (39)$$

Positivity of the denominator can be seen by taking the vector x'^μ as defined above and setting $\kappa = -x^2 > 0$. From $x'^2 < 0$ we obtain

$$x'^2 = (x^\mu - x^2(\theta v)^\mu)(x_\mu - x^2(\theta v)_\mu) = \underbrace{x^2}_{<0} (1 - 2x_\mu(\theta v)^\mu + x^2(\theta v)_\mu) < 0.$$

From the inequality it follows that the the denominator in (39) is positive and therefore one is left with proving

$$(x^1 - (\theta v)^1 x^2) > |(x^0 - (\theta v)^0 x^2)|. \quad (40)$$

By choosing $\kappa = -x^2$ this is exactly the inequality for the vector $x'^\mu \in W_1$. Therefore, the special conformal transformed coordinate is still in the right wedge. The proof that the special conformal transformations map the left wedge into the left wedge is analogous. \square

Proposition 4.10. *For $n = 4l + 1$, where $l \in \mathbb{N}_0$ the family of fields $\phi = \{\phi_W : W \in \mathcal{W}_0\}$ are wedge-local fields on the Bosonic Fockspace \mathcal{H}^+ .*

Proof. We first prove that the expression $[\alpha_{\theta v}(\phi(f)), \alpha_{-\theta u}(\phi(g))]$ vanishes for all $v, u \in \text{sp}U$ and for $f \in C_0^\infty(W_1)$, $g \in C_0^\infty(-W_1)$. By using proposition 4.8 it then follows, that the commutator $[\phi_{W_1}(f), \phi_{-W_1}(g)]$ vanishes.

$$\begin{aligned} [\alpha_{\theta v}(\phi(f)), \alpha_{-\theta u}(\phi(g))] &= (2\pi)^{-2(n+1)} \iint d^{n+1}x d^{n+1}y f(x)g(y) [\alpha_{\theta v}(\phi(x)), \alpha_{-\theta u}(\phi(y))] \\ &= (2\pi)^{-2(n+1)} \iint d^{n+1}x d^{n+1}y f(x)g(y) \sigma_{\theta v}(x)^{\frac{1-n}{2}} \sigma_{-\theta u}(y)^{\frac{1-n}{2}} [\phi(x_{\theta v}), \phi(y_{-\theta u})] = 0. \end{aligned}$$

In the last line we applied lemma 4.9 to prove that after the special conformal transformation, the support of the field ϕ_{W_1} stays in the right wedge and the support of the field ϕ_{-W_1} stays in the left wedge. Therefore, the supports of the fields are space-like separated, hence they commute. \square

Lemma 4.11. *In four dimensions a unitary representation for the whole conformal group, which gives the correct transformation law (18), exists for the fields $\phi_{\theta, K}(f)$ with $f \in C_0^\infty(W_1)$. The same holds for the field $\phi_{-\theta, K}(g)$ with $g \in C_0^\infty(-W_1)$.*

Proof. The problem with the absence of a unitary representation for the whole conformal group that gives the correct transformation law (18) is due to the absolute value of the scale factor $\sigma_b(x)$. Nevertheless, we showed in lemma 4.9 that the scale factor for a field localised in the right wedge is positive. The positivity of the scale factor in turn means that a unitary representation for the whole conformal group in four dimensions exists, [29].

$$\begin{aligned} \phi_{W_1}(f)\Psi &= \phi_{\theta, K}(f)\Psi = (2\pi)^{-4} \int d^4x f(x) \iint d^4v d^4u e^{-iuv} \alpha_{\theta v}(\phi(f))U(u)\Psi \\ &= (2\pi)^{-4} \int d^4x f(x) \iint d^4v d^4u e^{-iuv} \sigma_{\theta v}(x)^{-1} \phi(x_{\theta v})U(u)\Psi, \quad \Psi \in \mathcal{D}. \end{aligned}$$

For a quantum field defined on the left wedge the proof is done analogously. \square

Proposition 4.12. *For $n = 3$, the fields $\phi = \{\phi_W : W \in \mathcal{W}_0\}$ are wedge-local fields on the Bosonic Fockspace \mathcal{H}^+ .*

Proof. Due to the existence of a unitary representation shown in lemma 4.11 the deformed field can be defined for $n = 3$. Furthermore, by applying 4.8 one shows that the expression $[\alpha_{\theta v}(\phi(f)), \alpha_{-\theta u}(\phi(g))]$ vanishes for all $v, u \in \text{spU}$ and for $f \in C_0^\infty(W_1)$, $g \in C_0^\infty(-W_1)$.

$$\begin{aligned} [\alpha_{\theta v}(\phi(f)), \alpha_{-\theta u}(\phi(g))] &= (2\pi)^{-8} \iint d^4x d^4y f(x)g(y) [\alpha_{\theta v}(\phi(x)), \alpha_{-\theta u}(\phi(y))] \\ &= (2\pi)^{-8} \iint d^4x d^4y f(x)g(y) \sigma_{\theta v}(x)^{-1} \sigma_{-\theta u}(y)^{-1} [\phi(x_{\theta v}), \phi(y_{-\theta u})] = 0, \end{aligned}$$

Analogous to the proof of proposition (4.10) we use lemma 4.9 in the last line. \square

This is a very interesting result. The deformed case improves the representations such that one does not have to deal with representations of the covering of the conformal group.

5 NC Spacetime from special conformal operators

5.1 Moyal-Weyl

Within the framework of warped convolutions [7], the authors defined a deformed associative product in the following way.

Definition 5.1. *The associative deformed product \times_θ of A, B is defined as*

$$A \times_\theta B = (2\pi)^{-d} \iint d^d v d^d u e^{-iv u} \alpha_{\theta v}(A) \alpha_u(B). \quad (41)$$

Furthermore, the deformed commutator $[A \times_\theta^\circ B]$ of A, B is defined in the following way

$$[A \times_\theta^\circ B] := A \times_\theta B - B \times_\theta A. \quad (42)$$

The deformed product can be used to calculate the commutator of the coordinates. In the case of deformation with the momentum operator P_μ one obtains the following lemma.

Lemma 5.2. *Let the deformed product 5.1 be defined by the generator of translations P_μ . Then the deformed commutator 42 of the coordinates gives the Moyal-Weyl plane*

$$[x_\mu \times_\theta^\circ x_\nu] := x_\mu \times_\theta x_\nu - x_\nu \times_\theta x_\mu = -2i\theta_{\mu\nu}. \quad (43)$$

Proof. We first calculate the deformed product of the coordinates using definition 5.1.

$$\begin{aligned} x_\mu \times_\theta x_\nu &= (2\pi)^{-d} \iint d^d v d^d u e^{-iv u} \alpha_{\theta v}(x_\mu) \alpha_u(x_\nu) \\ &= (2\pi)^{-d} \iint d^d v d^d u e^{-iv u} (x_\mu + (\theta v)_\mu) (x_\nu + u_\nu) = x_\mu x_\nu - i\theta_{\mu\nu} \end{aligned}$$

In the last lines, we applied the adjoint action of the momentum operator P_μ on the coordinates, which induces a translation. The next step consists in calculating the deformed commutator of the coordinates. Due to the skew-symmetry of the deformation matrix θ , one obtains for the deformed commutator the Moyal-Weyl plane. \square

This result is not surprising. As already mentioned, in [12] a quantum field was defined on the Moyal-Weyl plane, which also can be obtained by using the momentum operator for deformation via warped convolutions. Therefore, it is only natural that the Moyal-Weyl plane appears for the deformed commutator of the coordinates. In the next section we will calculate the commutator of the coordinates by using the deformed product induced by the special conformal operators.

5.2 Nonconstant noncommutative spacetime

The main idea in this work is to use the special conformal operator to deform the free quantum field. We further proved that the deformed field satisfies some weakened covariance and locality properties. Now a natural question arises. What is the noncommutative spacetime that we obtain from the deformation with the special conformal operator? This question can be answered by calculating the deformed commutator of the coordinates.

$$[x_\mu \times_\theta x_\nu] = (2\pi)^{-d} \iint d^d v d^d u e^{-ivu} (\alpha_{\theta v}(x_\mu) \alpha_u(x_\nu) - \alpha_{\theta v}(x_\nu) \alpha_u(x_\mu)). \quad (44)$$

To calculate the term $\alpha_{\theta v}(x_\mu)$ we insert the generator K_μ as a differential operator defined in ([9]).

$$\alpha_{\theta v}(x_\mu) = \exp \left((\theta v)^\sigma \left(2x_\sigma x^\lambda \frac{\partial}{\partial x^\lambda} - x^2 \frac{\partial}{\partial x^\sigma} \right) \right) x_\mu =: \exp((\theta v)^\sigma K_\sigma(x)) x_\mu \quad (45)$$

We could use the transformation of the coordinates under the special conformal generators, but in that case we would not be able to solve the integral. The ansatz we follow in this work is to solve the integral, order by order. This will be done by performing a Taylor expansion of the exponentials.

Lemma 5.3. *Let the deformed product 5.1 be defined by the generator of special conformal transformations K_μ . Then the deformed commutator (44), up to third order in θ is given as follows*

$$[x_\mu \times_\theta x_\nu] = -2i\theta_{\mu\nu} x^4 - 4i((\theta x)_\mu x_\nu - (\theta x)_\nu x_\mu) x^2.$$

Proof. The deformed commutator gives the following

$$\begin{aligned} [x_\mu \times_\theta x_\nu] &= (2\pi)^{-d} \iint d^d v d^d u e^{-ivu} (\alpha_{\theta v}(x_\mu) \alpha_u(x_\nu) - \mu \leftrightarrow \nu) \\ &= (2\pi)^{-d} \iint d^d v d^d u e^{-ivu} \left(\sum_{k=0}^{\infty} \frac{1}{k!} \underbrace{(\theta v)^\sigma K_\sigma(x) \cdots (\theta v)^\rho K_\rho(x)}_k x_\mu \right) \\ &\quad \times \left(\sum_{l=0}^{\infty} \frac{1}{l!} \underbrace{(u)^\lambda K_\lambda(x) \cdots (u)^\tau K_\tau(x)}_l x_\nu - \mu \leftrightarrow \nu \right). \end{aligned}$$

There are two properties for the series that can be easily seen. First, the different orders between θv and u do not mix. The only terms which are not equal to zero are the terms of equal order. The vanishing of unequal orders between θv and u will be shown in the following calculation.

$$\begin{aligned} &\iint d^d v d^d u e^{-ivu} \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \frac{1}{k!l!} \underbrace{(\theta v)^\sigma \cdots (\theta v)^\rho}_k \underbrace{u^\lambda \cdots u^\tau}_l \left(\underbrace{K_\sigma(x) \cdots K_\rho(x)}_k x_\mu \right) \\ &\times \left(\underbrace{K_\lambda(x) \cdots K_\tau(x)}_l x_\nu \right) \end{aligned}$$

$$\begin{aligned}
&= \iint d^d v d^d u \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \frac{(-i)^k}{k!l!} \underbrace{\theta^\sigma \cdots \theta^\rho}_k \left(\underbrace{\frac{\partial}{\partial u_\kappa} \cdots \frac{\partial}{\partial u_\gamma}}_k e^{-ivu} \right) \underbrace{u^\lambda \cdots u^\tau}_l \\
&\times \left(\underbrace{K_\sigma(x) \cdots K_\rho(x)}_k x_\mu \right) \left(\underbrace{K_\lambda(x) \cdots K_\tau(x)}_l x_\nu \right) \\
&= \iint d^d v d^d u e^{-ivu} \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \frac{i^k}{k!l!} \underbrace{\theta^\sigma \cdots \theta^\rho}_k \left(\left(\underbrace{\frac{\partial}{\partial u_\kappa} \cdots \frac{\partial}{\partial u_\gamma}}_k \right) \underbrace{u^\lambda \cdots u^\tau}_l \right) \\
&\times \left(\underbrace{K_\sigma(x) \cdots K_\rho(x)}_k x_\mu \right) \left(\underbrace{K_\lambda(x) \cdots K_\tau(x)}_l x_\nu \right) \\
&= \sum_{k=0}^{\infty} \frac{(-i)^k}{k!} \underbrace{\theta^{\sigma\lambda} \cdots \theta^{\rho\tau}}_k \left(\underbrace{K_\sigma(x) \cdots K_\rho(x)}_k x_\mu \right) \left(\underbrace{K_\lambda(x) \cdots K_\tau(x)}_k x_\nu \right)
\end{aligned}$$

In the third line we performed a partial integration. The expression vanishes in the case $k > l$, because the differentials annihilate the polynomial in u . It also vanishes if $k < l$ because nonvanishing polynomials in u stay and the integral sets the polynomials zero. Furthermore, by using the symmetry of the x -dependent differential operators K , one solves the integral. It is important to note that the result of the deformed product between the coordinates, is exactly the same result one would encounter by using twist-deformation with the special conformal operators, [1].

The second observation is that polynomials in u, v that are even vanish due to the anti-symmetry of the commutator. This is shown in the following.

$$\begin{aligned}
&\iint d^d v d^d u e^{-ivu} \left(\underbrace{(\theta v)^\sigma K_\sigma(x) \cdots (\theta v)^\rho K_\rho(x)}_{2m} x_\mu \underbrace{(u)^\lambda K_\lambda(x) \cdots (u)^\tau K_\tau(x)}_{2m} x_\nu \right) \\
&- \iint d^d v d^d u e^{-ivu} \left(\underbrace{(\theta v)^\sigma K_\sigma(x) \cdots (\theta v)^\rho K_\rho(x)}_{2m} x_\nu \underbrace{(u)^\lambda K_\lambda(x) \cdots (u)^\tau K_\tau(x)}_{2m} x_\mu \right)
\end{aligned}$$

Where m is a natural number. In the second integral we perform the integration variable substitution $(v, u) \rightarrow (\theta^{-1}u, \theta v)$ and obtain

$$\begin{aligned}
&\iint d^d v d^d u e^{-ivu} \left(\underbrace{(\theta v)^\sigma K_\sigma(x) \cdots (\theta v)^\rho K_\rho(x)}_{2m} x_\mu \underbrace{(u)^\lambda K_\lambda(x) \cdots (u)^\tau K_\tau(x)}_{2m} x_\nu \right) \\
&- \iint d^d v d^d u e^{ivu} \left(\underbrace{(\theta v)^\sigma K_\sigma(x) \cdots (\theta v)^\rho K_\rho(x)}_{2m} x_\mu \underbrace{(u)^\lambda K_\lambda(x) \cdots (u)^\tau K_\tau(x)}_{2m} x_\nu \right).
\end{aligned}$$

After performing the integration variable substitution $u \rightarrow -u$ we obtain

$$\begin{aligned} & \iint d^d v d^d u e^{-iv u} \left(\underbrace{(\theta v)^\sigma K_\sigma(x) \cdots (\theta v)^\rho K_\rho(x)}_{2m} x_\mu \underbrace{(u)^\lambda K_\lambda(x) \cdots (u)^\tau K_\tau(x)}_{2m} x_\nu \right) \\ & - (-1)^{2m} \iint d^d v d^d u e^{-iv u} \left(\underbrace{(\theta v)^\sigma K_\sigma(x) \cdots (\theta v)^\rho K_\rho(x)}_{2m} x_\mu \underbrace{(u)^\lambda K_\lambda(x) \cdots (u)^\tau K_\tau(x)}_{2m} x_\nu \right) \\ & = 0. \end{aligned}$$

Therefore, the only terms that do not vanish are those of equal odd order in v and u . In the following we calculate the noncommutativity of the coordinates up to the second order and obtain

$$\begin{aligned} [x_\mu \times_\theta x_\nu] &= (2\pi)^{-d} \iint d^d v d^d u e^{-iv u} \left((\theta v)^\sigma (2x_\sigma x_\mu - x^2 \eta_{\sigma\mu}) (u)^\tau (2x_\tau x_\nu - x^2 \eta_{\tau\nu}) - \mu \leftrightarrow \nu \right) \\ &= -2i\theta_{\mu\nu} x^4 - 4i \left((\theta x)_\mu x_\nu - (\theta x)_\nu x_\mu \right) x^2 + \mathcal{O}(\theta^3). \end{aligned}$$

□

The deformed commutator of the coordinates shows that the deformation induced by the special conformal operators spans a nonconstant noncommutative spacetime. This is very interesting because the spacetime that we obtain is a curved noncommutative spacetime and the curvature of the noncommutative spacetime is induced by the special conformal operators. In the case of using the momentum operator, i.e. the generator of translations in Minkowski, for the deformation one obtains a flat noncommutative spacetime. The special conformal operators induce a conformally flat spacetime on Minkowski and therefore, one obtains a conformally flat noncommutative spacetime when deforming with K_μ . Some examples of a nonconstant noncommutative spacetime exist in literature, where the highest order of noncommutativity known is the so called quantum space structure, [20, 30, 31]. The quantum space structure has an x -polynomial dependence up to second order.

5.3 Generalisation of the deformation

The deformation of an operator by either using the momentum operator P_μ or the special conformal operator K_μ can be written in a general form. The generalisation can be accomplished by using a linear combination of generators of the pseudo-orthogonal group $SO(2, d)$. First, we redefine the operators P^μ and K^ν in the following way

$$\tilde{P}^\mu := \begin{pmatrix} \lambda' P^0 \\ \lambda' P^1 \\ \eta' P^2 \\ \eta' P^3 \end{pmatrix}, \quad \tilde{K}^\mu := \begin{pmatrix} \lambda K^0 \\ \lambda K^1 \\ \eta K^2 \\ \eta K^3 \end{pmatrix}, \quad (46)$$

where $\lambda', \lambda \in \mathbb{R}^+$ and $\eta', \eta \in \mathbb{R}$. In the next step we redefine the Lorentz generators $J^{4,\mu}$, $J^{5,\mu}$, $J^{\pm,\mu}$ and the skew-symmetric matrix θ as follows

$$\tilde{J}^{4,\mu} := \frac{1}{2} \left(\tilde{P}^\mu - \tilde{K}^\mu \right), \quad \tilde{J}^{5,\mu} := \frac{1}{2} \left(\tilde{P}^\mu + \tilde{K}^\mu \right) \quad (47)$$

$$\tilde{J}^{\pm,\mu} := \tilde{J}^{5,\mu} \pm \tilde{J}^{4,\mu}, \quad (48)$$

$$\tilde{\theta} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}. \quad (49)$$

Definition 5.4. Let $\tilde{\theta}$ be a real skew-symmetric matrix given in (49) and let $A \in C^\infty$. Then the generalized warped convolutions, i.e. the deformation of A denoted as A_θ^\pm is defined as follows

$$A_\theta^\pm \Psi := (2\pi)^{-d} \iint d^4 y d^4 k e^{-iy_\mu k^\mu} U^\pm(\theta y) A U^\pm(\theta y) U^\pm(k) \Psi, \quad \Psi \in \mathcal{D}, \quad (50)$$

where the unitary operator $U^\pm(k)$ is defined as $U^\pm(k) := \exp\left(ik^\mu \tilde{J}_\mu^\pm\right)$.

The generalisation of the deformation is interesting because it is obtained as a linear combination of generators of $SO(2,4)$. By choosing the plus sign, one obtains the Moyal-Weyl case and by choosing the minus sign one gets the special conformal model introduced in this work.

6 Conclusion and outlook

In this work we deformed a quantum field theory with the special conformal operator K_μ , using the warped convolutions. To proceed with the deformation, self-adjointness of the generator K_μ is proven in order to obtain a strongly continuous automorphism of the group \mathbb{R}^n . The proof of self-adjointness was done rigorously in [29] and is sketched in Sec. 3. Therefore, we were able to define the deformation of the scalar field with the special conformal operators. We further proved that the deformed quantum field satisfies the Wightman axioms, except for the covariance and locality.

The homomorphism $Q(\Lambda W)$, defined in [12] was used in this work to define the map from the deformed field ϕ_θ to a field defined on the wedge ϕ_W . Furthermore, it was proven that the field ϕ_W transforms as a wedge-covariant field under the adjoint action of the Lorentz group. Wedge-locality for the field ϕ_W was shown in $d = 4l + 2, l \in \mathbb{N}_0$ dimensions. In 4 dimensions one usually has a problem with the existence of a unitary representation for the whole conformal group. The absence of a unitary representation is due to the absolute value of the scale factor induced by the special conformal transformations. We circumvented the problem by proving positivity of the scale factor. Positivity was proven by using the properties of the wedge and the spectrum condition of the special conformal operator.

The deformed product defined in [7] is used to understand the noncommutative spacetime being induced by using the special conformal operators. The deformed product is used to calculate the commutator of the coordinates. We first proved that the formula obtained by solving the integral is known in literature as twist deformation. Furthermore, we discovered that the noncommutative spacetime obtained in this manner is a nonconstant noncommutative spacetime which seems to be a new result.

To calculate the S-matrix in the current framework we have to use the concept of the temperate polarization-free generators defined in [4]. The concept can be used for fields deformed with the momentum operator, but for the special conformal operator we still have to work out some technical subtleties. This will be done in a further work.

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