On the Net of von Neumann Algebras
Associated with a Wedge
and Remarks on the Connes-Type of Local Algebras

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Abstract:

The wedge in Minkowski space has the property that its space-like complement of this set coincides with the reflection at the origin of this set. This implies that the commutant of the von Neumann algebra associated with the wedge coincides with the algebra associated with the opposite set. This geometric symmetry implies symmetries for Tomita's modular theory also.

If one defines sub-algebras of the double cones or cylinders by intersecting the algebras of the shifted wedges, one can re-construct the algebras of the larger double-cones or the wedge with help of either the translations or the modular group of the wedge-algebra.

The symmetry of the wedge and its algebra implies by simple arguments the von Neumann algebra of the wedge is of type III. By a careful look at Connes' definition of the classification of the type III-algebras we will show that this algebra is of Connes-type III_1. By the same method we will find that also the von Neumann algebras of double-cones are of Connes-type III_1.

1. Introduction-

In earlier papers I investigated the sub-algebras having the same cyclic and separating vector [1]. I started this investigation since I hoped that this method could be used for local quantum field theory, where all the local algebras have the same cyclic and separating vector also. But it turned out that one obtains by this procedure too many algebras in order that it could be useful for physics. Moreover, these algebras contain algebras of different Connes-von Neumann types. This is due to the split-property [2]. One even does not know how to select algebras with the same von Neumann type.

Discussing this situation with D. Buchholz, he suggested to start with the algebras of wedges and to derive from this all the local algebras. A guide to such enterprise would be the paper of G. Lechner [3] and others who have solved this problem for the two-dimensional case. We will look at this problem for the higher-dimensional situation.

As usual I started with a separable Hilbert space $\mathcal{H}$ on which there exists a unitary representation of the translation group of $\mathbb{R}^d$ fulfilling the spectrum-condition and which possesses a unique invariant vector Ω. In section 2 the wedge-algebra will be defined. In addition it will be assumed that Ω is cyclic and separating for the wedge-algebra. Using this input we will define the algebras for the space-like cylinders and the double-cones. The last algebra will be defined without Lorentz- or rotation-transformations. The only input is the geometrical structure of the Minkowski space. Having defined these algebras we will investigate their properties, in particular the Reeh-Schlieder theorem [4] of these algebras which implies that Ω is also cyclic and separating for these algebras.

Section 3 we start with the algebra of a cylinder or with that of a double cone, and show how to re-construct the algebra of larger cylinders or double cones. With the same method the algebra of the wedge can be constructed. To do this we will use the half-sided translation [5] and we will use techniques of analytic functions of several complex variables, which can be used because of the spectrum condition for the translation. Out of this method we take the double-cone theorem [6],[7]. Finally we will look in section 3 at the centre of the wedge algebra and show that it coincides with that of the global algebra.

In section 4 we look at the type question of local algebras. Although this has already been solved by Fredenhagen [8], using a result of R. Longo [9], who derived the Connes-von Neumann type for the wedge. I thought it would be useful to have a new look at this problem and to develop new techniques. This is desirable since the paper of Fredenhagen [8] uses additional properties. For our investigation we develop new methods to find the invariant S by starting directly with Connes' definition [10] of his invariant S. We will show that as well the algebras of the space-like cylinder as that of the double-cones have the Connes-von Neumann type III_1. I hope that the results of section 2, 3, and 4 are useful for the construction of interacting quantum fields in higher dimension. A similar result has been obtained by Araki [11] but by different methods.

1.1. Assumptions and notations:

a) Let $\mathcal{H}$ be a separable Hilbert space. Assume on $\mathcal{H}$ exists a continuous unitary representation of the translation group $T(a)$ of the d-dimensional Minkowski space.

- α Moreover, assume there exists a unique unit-vector $\Omega \in \mathcal{H}$ with the property $T(a)\Omega = \Omega, \forall a \in \mathbb{R}^d$.
- β In addition assume that the spectrum of the translation group $T(a)$ is contained in the forward light-cone V^+.

b) Let $\mathcal{M}$ be a von Neumann algebra acting on $\mathcal{H}$. We say $\Omega \in \mathcal{H}$ is cyclic and separating for $\mathcal{M}$ if $\mathcal{M}\Omega$ and $\mathcal{M}'\Omega$ are dense in $\mathcal{H}$. The algebra $\mathcal{M}'$ denotes the commutant of $\mathcal{M}$. In this situation exists by the Tomita-Takesaki theory [12,13] a modular operator Δ which is non-negative and a modular conjugation J fulfilling

$$\begin{aligned} \Delta\Omega = \Omega,\ J\Omega = \Omega, \\ \mathrm{Ad}\,\Delta^{\mathrm{i}t}\mathcal{M} = \mathcal{M},\ J\mathcal{M}J = \mathcal{M}' \\ J\Delta^{\mathrm{i}t}J = \Delta^{\mathrm{i}t},\ J\Delta^{\frac{1}{2}}A\Omega = A^*\Omega, \forall A \in \mathcal{M} \end{aligned}$$

c) Let $U(s)$ be a one-parametric unitary group. We say $U(s)$ is a $(\pm)$-half-sided translation for $\mathcal{M}$ if the following conditions are fulfilled:

$$\begin{aligned} U(s)\Omega = \Omega, \\ U(s) \text{ has } \quad \text{a positive spectrum} \\ \mathrm{Ad}\,U(s)\mathcal{M} \subset \mathcal{M}, \quad \text{for}(\pm)s \in \mathbb{R}^+ \end{aligned}$$

If $U(s)$ fulfils these conditions then there exist between the modular group of $\mathcal{M}$ and $U(s)$ the following relations:

$$\begin{aligned} \mathrm{Ad}\,\Delta^{\mathrm{i}t} U(s) &= U(\mathrm{e}^{(\mp)2\pi t} s), \\ J U(s) J &= U(-s). \end{aligned} \tag{1.1}$$

These results can be found in [5].

d) Denote by V^+ the forward light-cone.

α Let $\ell_1 \neq \ell_2$ be two light-rays belonging to ∂V^+ then the wedge $W(\ell_1, \ell_2)$ is defined by the formula:

$$W(\ell_1, \ell_2) = \{a_1 \ell_1 - a_2 \ell_2 + \hat{a},\ \ a_1, a_2 > 0, \hat{a} \perp (\ell_1, \ell_2)\}$$

β If $t_0 \in V^+$ is a fixed time-like vector with $t_0^2 = 1$ and $\ell \in \partial V^+$ then we denote by ℓ' the light-like vector in the intersection of ∂V^+ with the two-plane spanned by ℓ and t_0.

Let the space-like vector $a_1, a_1^2 = -1$ belong to the two-plane spanned by (t_0, ℓ). In this case we set

$$a^+ = t_0 + a_1, \quad a^- = t_0 - a_1.$$

Now we identify ℓ with a^+ and ℓ' with a^-. In this special situation the two-dimensional wedge $W_2 = W(\ell_1, \ell_2) \cap \mathbb{R}^2(t_0, a_1)$ can be written as

$$W_2 = \{t, a;\ |t| \leq a;\ t = \alpha_0 t_0, a = \alpha_1 a_1\}.$$

γ Let $\mathcal{M}(W(\ell_1, \ell_2))$ be the algebra associated with $W(\ell_1, \ell_2)$, then we identify the commutant with the following algebra

$$\mathcal{M}(W(\ell_1, \ell_2))' = \mathcal{M}(W(\ell_2, \ell_1)),$$

but only if Ω is cyclic and separating for $\mathcal{M}(W(\ell_1, \ell_2))$.

Locality and the Reeh-Schlieder theorem [4] imply that Ω is cyclic and separating for $\mathcal{M}(W_2(\ell_1, \ell_2))$ and for sub-algebras which are defined by intersections of shifted wedge-algebras, as double-cones and cylinders.

Next we look at special situations described in 1.1. and some applications. We start with the two-dimensional wedge.

1.2. Modular group of the wedge algebra in two dimensions and the translation group

Let $\mathcal{M}(W_2)$ be the von Neumann algebra associated with W_2. If Ω is cyclic and separating for $\mathcal{M}(W_2)$, then $T(\lambda^+ a^+)$ is a $(+)$-half-sided translation for $\mathcal{M}(W_2)$ and $T(\lambda^- a^-)$ is a $(-)$-half-sided translation for the same algebra. For simpler writing we set Δ for the modular operator of $\mathcal{M}(W_2)$. Now we obtain:

$$\begin{aligned} \mathrm{Ad}\,\Delta^{\mathrm{i}t} T(\lambda^+ a^+) &= T(\mathrm{e}^{-2\pi t}\lambda^+ a^+), \\ \mathrm{Ad}\,\Delta^{\mathrm{i}t} T(\lambda^- a^-) &= T(\mathrm{e}^{2\pi t}\lambda^- a^-). \end{aligned} \tag{1.2}$$

Notice that the sign in the exponential is opposite to the sign of the half-sided translation. Now let a be a vector in the two-dimensional wedge, then the two equations imply:

$$\mathrm{Ad}\,\Delta^{\mathrm{i}t} T(a) = T(\Lambda_2(t)a)$$

with $\Lambda_2(t)$ a Lorentz transformation

$$\Lambda_2(t) = \begin{pmatrix} \cosh t & -\sinh t \\ -\sinh t & \cosh t \end{pmatrix}. \tag{1.2.a}$$

If we look at the opposite wedge $W' = \{(t,a), |t| \leq -a\}$ with t a multiple of t_0 and a a multiple of a_1. Since the modular group of the commutant coincides with that of the algebra, we obtain for $a \in W'$ the same as for W:

$$\begin{aligned} \mathrm{Ad}\,\Delta^{\mathrm{i}t} T(\beta a^+) &= T(\mathrm{e}^{-2\pi t}\beta a^+) \\ \mathrm{Ad}\,\Delta^{\mathrm{i}t} T(\alpha a^-) &= T(\mathrm{e}^{2\pi t}\alpha a^-) \end{aligned}. \tag{1.3}$$

This implies for $a \in W_2'$ again

$$\mathrm{Ad}\,\Delta^{\mathrm{i}t} T(a) = T(\Lambda_2(t)a).$$

Remark:

New features are obtained for the forward- and backward-light-cone, provided we are dealing with a massles theory, where the forward- cone is the support of an algebra. In this situation we obtain from (1.2) and (1.3) the same sign for a^+ and a^-.

$$\mathrm{Ad}\,\Delta^{\mathrm{i}t}(\lambda a^{\pm}) = T(\mathrm{e}^{-2\pi t}\lambda a^{\pm}).$$

This implies for $a \in V^+$:

$$\mathrm{Ad}\,\Delta^{\mathrm{i}t} T(\lambda a) = T(\mathrm{e}^{-2\pi t}\lambda a).$$

This is a dilatation, precisely for positive t a contraction and for negative t an expansion. For V^- we find:

$$\mathrm{Ad}\,\Delta^{\mathrm{i}t} T(\lambda a) = T(\mathrm{e}^{2\pi t}\lambda a).$$

Usually this result is not connected with an algebra. The only exception is the case of massless fields. See[14].

The result (1.2.a) will be used in the next section.

1.3. One half-sided translation for different algebras

We leave the two-dimensional situation and go to the higher-dimensional case. In 1.1.$d.\alpha$ we introduced for $\ell_1, \ell_2 \in \partial V^+$ the wedge $W(\ell_1, \ell_2)$. If we keep ℓ_1 fixed and vary $\ell_2 \neq \ell_1$ then we obtain a family of wedges $W(\ell_1, \ell_i)$ such that $T(\lambda \ell_1)$ is a half-sided translation for every of the algebras $\mathcal{M}(W(\ell_1, \ell_i))$. But these algebras are not the only one. If we keep ℓ_1 fixed, then $\mathcal{M}(\bigcap_{i=1}^{n} \mathcal{M}(W(\ell_1, \ell_i))$ has again $T(\lambda\ell_1)$ as half-sided translation. In every of these cases we obtain $\mathrm{Ad}\, \Delta^{\mathrm{i}t} T(\lambda \ell_1) = T(\mathrm{e}^{-2\pi t} \lambda \ell_1)$, where Δ is the modular operator of the mentioned algebras. Let now Δ_1, Δ_2 be the modular operators of two different of these algebras, then we get that $\Delta_1^{\mathrm{i}t} \Delta_2^{-\mathrm{i}t}$ commutes with the translations $T(\lambda\ell_1)$. Whether or not these unitary groups generate the whole commutant of $T(\lambda \ell_1)$ is not known. This is due to the fact that a modular group does not determine the algebra. In case we are dealing with a Lorentz covariant theory the two algebras $\mathcal{M}(W(\ell_1, \ell_2))$ and $\mathcal{M}(W(\ell_1, \ell_3))$ are connected by a Lorentz transformation belonging to the fixed group of ℓ_1.

2. Construction of the local net from the wedge algebra

We start to list the assumptions for this section.

2.1 Assumptions and notations:

1) Let $t_0, t_0^2 = 1$ be a chosen fixed time-like direction and $a_1, (t_0, a_1) = 0, a_1^2 = -1$ be a fixed space-like direction. Denote by $W_2 = \{a = \alpha_0 t_0 + \alpha_1 a_1\}$ with a contained in the two-space generated by t_0 and a_1 and $|\alpha_0| \leq \alpha_1$. If $d > 2$ then we set $W = \{W_2 + \hat{a}\}, \tilde{a} \perp W_2$.
2) By $\mathcal{M}(W)$ we denote a von Neumann algebra acting on $\mathcal{H}$ with the property $\mathrm{Ad}\, T(a) \mathcal{M}(W) \subset \mathcal{M}(W), \forall a \in W$. Moreover, Ω shall be cyclic and separating for $\mathcal{M}(W)$. $\mathcal{M}(W')$ denotes the commutant of $\mathcal{M}(W)$.
2.a) Moreover, we assume that $\mathrm{Ad}\, T(-\lambda a_1) \mathcal{M}(W) \cap \mathrm{Ad}\, T(\lambda a_1) \mathcal{M}(W)'$ is a proper algebra for every $\lambda > 0$.
2.b) If the dimension is larger than 2 we require that $\bigcap_{r\in\mathcal{R}}$ applied to the sets described in 2.a) is not empty (non-empty in 2.a and 2.b) means that the intersection of the corresponding algebras is a non-trivial algebra. $\mathcal{R}$ stands for the rotation-group around the time-axis. The expression $\bigcap_{r\in\mathcal{R}}$ stands for the definition of the double-cone given in assumption (3,2).

Although we give in (2.1) a definition of the double-cone-algebra without using the rotational invariance, we could have used the rotations since the invariance property has been shown in [11].

3,1) We define the algebra of the cylinder ${}^0Z_\lambda$ by the equation

$$\mathcal{M}({}^0Z_\lambda) = [\mathrm{Ad}\, T(-\lambda a_1 \mathcal{M}(W))] \bigcap [\mathrm{Ad}\, T(\lambda a_1) \mathcal{M}(W'))]$$

The upper index zero in front of Z or D indicates that the centre of these sets is located at zero.

3,2) If the dimension $d > 2$, then we keep t_0 fixed and vary a_1 in the boundary of V^+. By this we obtain a family of wedges $W(a_1^i)$ and a family of different cylinders and their algebras $\mathcal{M}({}^0Z(a_1^i, \lambda))$. Keeping λ fixed, we define the algebra of the double cone

$$\mathcal{M}({}^0D(\lambda)) = \bigcap_{a_1^i \in \partial V^+} \mathcal{M}({}^0Z(a_1^i, \lambda)). \tag{2.1}$$

Having introduced our notation we can start with the investigation, where we use the assumptions and notations of 2.1. The first result is concerned with properties of the cylinder.

4) Since we will use increasing families of von Neumann algebras, we will assume continuity from inside for algebras based on increasing sets of the Minkowski space.

2.2. Lemma:

Denote by ${}^0D_2(\lambda)$ the restriction of ${}^0D(\lambda)$ to the two-space generated by (t_0, a_1). Let $\lambda_2 > \lambda_1$ then we get ${}^0Z(\lambda_1) \subset {}^0Z(\lambda_2)$ and ${}^0D(\lambda_1) \subset {}^0D(\lambda_2)$.

$$\bigvee_{b \in {}^0D_2(\lambda_2 - \lambda_1)} \operatorname{Ad} T(b)\mathcal{M}({}^0Z_{\lambda_1}) = \mathcal{M}({}^0Z_{\lambda_2}). \tag{2.2}$$

$\vee\{\mathcal{M}({}^0Z_{\lambda_i})\}$ denotes the von Neumann algebra generated by all $\mathcal{M}({}^0Z_{\lambda_i})$.

Proof: The time-translation along the middle axis of ${}^0D(\lambda_2 - \lambda_1)$ is:

$$\bigvee_{|\mu| \leq \frac{(\lambda_2 - \lambda_1)}{2}} T(\mu){}^0Z(\lambda_1). \tag{2.3}$$

This domain contains a neighbourhood of time-axis between $-\lambda_2$ and $+\lambda_2$. Using the double-cone theorem [6,7] we get as domain $Z(\lambda_2)$. For a different proof see lemma 2.3. q.e.d.

Next we go to double cones and obtain

2.3. Lemma:

Let $\lambda_1 < \lambda_2$ then holds:

$$\bigvee_{b \in {}^0D(\lambda_2 - \lambda_1)} \operatorname{Ad} T(b)\mathcal{M}({}^0D_{\lambda_1}) = \mathcal{M}({}^0D_{\lambda_2}). \tag{2.4}$$

Proof: Eq. (2.4) does not hold only for the standard wedge, but also for all other wedges with different $a \in \partial V^+$. With this notation we want to show

$$\bigvee_{b \in {}^0D(\lambda_2 - \lambda_1)} \operatorname{Ad} T(b)\mathcal{M}({}^0D(\lambda_1)) = \mathcal{M}({}^0D(\lambda_2)).$$

Applying Eq. (2.1) to the left-hand side of (2.2) we obtain:

$$\mathcal{M}(D(\lambda_2)) = \bigcap_{a\in\partial V^+} \bigvee_{b\in{}^0D(\lambda_2-\lambda_1)} \mathrm{Ad}\, T(b)\mathcal{M}({}^0Z(a,\lambda_1)).$$

Performing the intersection find:

$$\bigvee_{b\in{}^0D(\lambda_2-\lambda_1} \mathrm{Ad}\, T(b)\mathcal{M}({}^0D(\lambda_1)).$$

This formula tells us that we shall enlarge ${}^0D(\lambda_1)$ by the double cone ${}^pD(\lambda_2-\lambda_1)$. The upper index p indicates that p is the centre of the double cone. Notice that the double cone $D(\lambda)$ can be written as $(|t|-|a|) < \lambda$. This gives in our situation $((|t|-|a|) < \lambda_1) + ((|t|-|a|) < |t|(\lambda_1 + (\lambda_2-\lambda_1))) = D(\lambda_2)$. This means the formula where the union is interchanged with the intersection gives also $\mathcal{M}({}^0D(\lambda_2))$. q.e.d.

For the next result we need some notations:
The cylinders ${}^0Z$ and double-cones ${}^0D$ have their centre at the origin. In the future we need cylinders and double-cones sitting in the corner of the wedge. Therefore, we set

$$\begin{aligned} Z(\lambda) =&\mathrm{Ad}\, T(\lambda)^0Z(\lambda), \\ D(\lambda) =&\mathrm{Ad}\, T(\lambda)^0D(\lambda), \end{aligned}$$

and we have dropped the direction a in the cylinder.

2.4. Corollary:

The algebra of the wedge can be obtained by the following manner

$$\begin{aligned} \bigvee_{\lambda>0} \mathcal{M}(Z(\lambda)) =&\mathcal{M}(W), \\ \bigvee_{\lambda>0} \mathcal{M}(D(\lambda)) =&\mathcal{M}(W). \end{aligned}$$

Proof:

We start with the cylinders $Z(\lambda)$. The commutant of $\mathcal{M}(Z(\lambda))$ consists of two wedges:

$$\mathcal{M}'(Z(\lambda)) = \mathcal{M}'(W) \cup T(2\lambda)\mathcal{M}(W),$$

going with $\lambda \to \infty$ we obtain the first result. For the commutant of the double-cone algebra $\mathcal{M}'(D(\lambda))$ we obtain the union of the wedge algebras $T(2\lambda a_1)\mathcal{M}(W)$, rotated about the point (λa_1), i.e.,

$$\mathcal{M}'(D(\lambda)) = T(\lambda a_1) \bigvee_{r\in\mathcal{R}} R(r)T(\lambda a_1)\mathcal{M}(W).$$

Let h be the distance from the plane $\lambda a_1 = 0$. Now we look at the intersection of the hyperplane $a_1 = h$ with the boundary of $D(\lambda)$ under the assumption $h < \lambda$, then these points have from the a_1-axis the distance $\sqrt{\lambda^2-(\lambda-h)^2}$, for $0<h<\lambda$ and $\sqrt{\lambda^2-(h-\lambda)^2}$ for $\lambda<h<2\lambda$. For $\lambda\to\infty$ these points tend to infinity and therefore $D(\lambda)$ tends to the wedge. q.e.d.

3. Consequences of half-sided translations

Recall a half-sided translation for a von Neumann algebra $\mathcal{M}$ with cyclic and separating vector Ω, and a group $U(t)$ of $\mathcal{M}$, such that $U(t)\Omega=\Omega$ and $\mathrm{Ad}\,U(t)\mathcal{M}\subset\mathcal{M}$ for either $t\geq 0$ or $t\leq 0$. In the first case one speaks about +half-sided translations and in the other case about −half-sided translations. In case of the wedge we set $a^{\pm}=(a_1\pm a_0)$. Then the standard translations $T(a^{\pm})$ fulfil the conditions for $\mathcal{M}(W)$. Between half-sided translations of $\mathcal{M}$ and the modular group of $\mathcal{M}$ exists a remarkable relation:

$$\mathrm{Ad}\,\Delta_{\mathcal{M}}^{\mathrm{i}s}(T(t))=T(\mathrm{e}^{\mp 2\pi s}t).$$

Here the minus-sign in the exponent holds for +half-sided translations and the other sign for −half-sided translations. For the wedge algebra exist both kinds of half-sided translations, therefore, we introduce the following notation:

$$\Lambda_2(t)=\begin{pmatrix}\cosh(2\pi t) & -\sinh(2\pi t)\\ -\sinh(2\pi t) & \cosh(2\pi t)\end{pmatrix}. \tag{3.1}$$

The lower index 2 indicates that this is the transformation of the 2-plane generated by (a_1,t_0). All other components are kept fixed. For higher dimensions we write:

$$\Lambda(t)(a,\hat{a})=(\Lambda_2 a,\hat{a}).$$

Applied to the wedge we obtain:

$$\mathrm{Ad}\,\Delta^{\mathrm{i}t}(T(a,\hat{a}))=T(\Lambda(t)(a,\hat{a})),\ (a,\hat{a})\in W. \tag{3.2}$$

See[16] That the modular group of the wedge-algebra coincides with the Lorentz-boost has been shown first by Bisognano and Wichmamm [17].

For the application of the last result we use the notations introduced at the end of the last section.

3.1. Theorem:
1) Let $\lambda_2>\lambda_1$ then there exists $t(\lambda_2,\lambda_1)$ with

$$\mathrm{Ad}\,\Delta_W^{\mathrm{i}t}\mathcal{M}(Z(\lambda_1))\subset\mathcal{M}(Z(\lambda_2)),\ \text{for } |t|\leq t(\lambda_2,\lambda_1),\ \text{with } t(\lambda_2,\lambda_1)=\frac{1}{2\pi}\log\frac{\lambda_2}{\lambda_1}.$$

This value means exactly that for $|t|>t(\lambda_2,\lambda_1)$ the transformed set is no longer contained in $\mathcal{M}(Z(\lambda_2))$.
2) Now holds:

$$\bigvee_{|t|\leq t(\lambda_2,\lambda_1)}\mathrm{Ad}\,\Delta_W^{\mathrm{i}t}\mathcal{M}(Z(\lambda_1))=\mathcal{M}(Z(\lambda_2)).$$

Proof: 1) The value of $t(\lambda_2, \lambda_1)$ is determined by the tips of the transformed double cone which has at most the value λ_2. This leads to the relation $e^{2\pi t}\lambda_1 = \lambda_2$.
2) Let G be the domain in W below the space-like hyperboloid of mass $2\lambda_1$ which is sitting in $D_2(\lambda_2)$. Moreover, let D_s be a small double-cone of radius $\mu < \lambda_1$ and let G_1 be the set of all b such that $T(b)D_s \subset G$. Choose two vectors $\psi_1, \psi_2 \in \mathcal{H}$ which are entire analytic for $T(x)$ and define the two functions

$$\begin{array}{l} F^+(x) = (\psi_1, B\{\mathrm{Ad}\, T(x)(A)\}\psi_2) \\ F^-(x) = (\psi_1, \{\mathrm{Ad}\, T(x)(A)\}B\psi_2) \end{array},$$

with B an operator commuting with $\mathcal{M}(Z(G))$ and A an operator belonging to $\mathcal{M}(Z(D_s))$. Then $F^+(x)$ has an analytic extension into the forward tube T^+ and $F^-(x)$ has an analytic extension into the backward tube T^-. In addition one has $F^+(x) = F^-(x)$ for $x \in G_1$. Using the double-cone theorem (see [6,7]) one finds $F^+(x) = F^-(x)$ for $x \in Z(\lambda_2 - \mu)$. Taking the limit $\mu \to 0$ one finds B commutes with $\mathcal{M}(Z(\lambda_2))$. This shows the theorem. q.e.d.

3.2. Corollary:
$\lambda_1 > 0$, *then holds*

$$\bigvee_{|t|>0} \mathrm{Ad}\, \Delta_W^{it} \mathcal{M}(Z(\lambda_1)) = \mathcal{M}(W).$$

Proof: For every $\lambda_2 > \lambda_1$ we obtain from Thm. 3.1

$$\bigvee_{|t|\le t(\lambda_2,\lambda_1)} \mathrm{Ad}\, \Delta_W^{it} \mathcal{M}(Z(\lambda_1)) = \mathcal{M}(Z(\lambda_2)).$$

Taking the limit $\lambda_2 \to \infty$ we obtain the result by lemma 2.4. q.e.d.
After this we turn to the structure analysis of algebras by using:

3.3. Theorem:
The centre of the algebra of the wedge $\mathcal{C}(\mathcal{M}(W))$ coincides with the centre of the global algebra $\bigvee_{b\in\mathbb{R}^d} \mathrm{Ad}\, T(b)\mathcal{M}(W)$.

Proof: $T(\lambda a^+)$ is a +half-sided translation for $\mathcal{M}(W)$. Therefore, we know from [18] Thm.2.4. that $\mathcal{C}(\mathcal{M}(W))$ is point-wise invariant under the action of $T(\lambda a^+)$. Let ψ be a vector entire analytic for $T(\lambda a^+)$ and $C \in \mathcal{C}(\mathcal{M}(W))$ and $A \in \mathcal{M}(W)$. Then vector function $\mathbf{F}(\lambda) = [\mathrm{Ad}\, T(\lambda a^+)A, C]\psi$ has an analytic continuation into the upper half-plane. Moreover, $\mathbf{F}(\lambda)$ vanishes for $\lambda > 0$ because of the condition for +half-sided translations. Hence $\mathbf{F}(\lambda)$ vanishes for all $\lambda \in \mathbb{R}$. This means C commutes with all algebras located in the half-space below the plane, characterized by $\{\lambda a^+\}$.

Since $\mathcal{M}(W)$ is also invariant under −half-sided translation by $T(\lambda a^-)$ all the arguments we used for +half-sided translations, after suitable adaptation, can be used for this case. Hence $C \in \mathcal{C}(\lambda) \forall \lambda \in \mathbb{R})$. This means $C \in \mathcal{C}(\mathcal{M}(W))$ commute with all algebras located in the half-space above the plane characterized by $\{\lambda a^-\}$. This means C commutes with all A located everywhere, except for W. Since C commutes also with $\mathcal{M}(W)$ it commutes with all operators. (If there exists operators located on the boundary of W, then they can be included into the commutant of $\mathcal{C}(\mathcal{M}(W))$ with help of the double-cone theorem.) q.e.d.

In corollary 3.2. we have constructed larger cylinders from smaller ones by using the modular group of the wedge-algebra. This method can be used also for double cones.

The algebra $\vee_{-|t| \le t(\lambda_2,\lambda_1)} \operatorname{Ad} \Delta_W^{it} \mathcal{M}(D_{\lambda_1})$ presents the algebra of a set, which ends at the boundary of D_{λ_2}. Applying to this the double-cone theorem we obtain $D(\lambda_2)$. Collecting the results of our discussion we obtain:

3.4. Theorem:

The algebra

$$\bigvee_{|t| \le t(\lambda_2,\lambda_1)} \operatorname{Ad} \Delta_W^{it} \mathcal{M}(D_{\lambda_1}) \tag{3.4}$$

coincides with $\mathcal{M}(D_{\lambda_2})$.

If in Eq. (3.4) is no restriction for t *we obtain the algebra of the wedge.*

At the end of this section we want to look at the three-dimensional group generated be the two dimensional translations of $\mathbb{R}^2$ and the modular group of the wedge-algebra: $(t,a), t \in \mathbb{R}, a \in \mathbb{R}^2$,

$$(t_1,a_1)(t_2,a_2) = (t_1+t_2, \Lambda_2(t_2)a_1 + a_2). \tag{3.5}$$

The investigation of this group is best done in form of 3×3 matrices:

$$\begin{pmatrix} 1 & 0 & a_1 \\ 0 & 1 & a_0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cosh 2\pi t & -\sinh 2\pi t & 0 \\ -\sinh 2\pi t & \cosh 2\pi t & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \cosh 2\pi t & -\sinh 2\pi t & a_1 \\ -\sinh 2\pi t & \cosh 2\pi t & a_0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{3.6.a}$$

For the investigation of this group, it is better to introduce light-cone coordinates ua^+, va^-. With this (3.6.a) reads:

$$\begin{pmatrix} 1 & 0 & u \\ 0 & 1 & v \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} e^{-2\pi t} & 0 & 0 \\ 0 & e^{2\pi t} & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} e^{-2\pi t} & 0 & u \\ 0 & e^{2\pi t} & v \\ 0 & 0 & 1 \end{pmatrix}. \tag{3.6}$$

This is the product of the two-dimensional translation group and the one-parametric modular group. The one-dimensional sub-groups can easily be determined. One obtains

$$\begin{pmatrix} e^{ar} & 0 & \frac{b}{a}(e^{ar}-1) \\ 0 & e^{-ar} & -\frac{c}{a}(e^{-ar}-1) \\ 0 & 0 & 1 \end{pmatrix}. \tag{3.7}$$

The modular group is obtained for $a=-2\pi$ and $b,c=0$, while the two translation groups are obtained for $a=0, br=u$ and $c=0$ and the other translation for $a=0, cr=v$ and $b=0$.
A group of special interest is obtained for $a=-2\pi, b=-2\pi u, c=0$, which reads in matrix form

$$\begin{pmatrix} \mathrm{e}^{-2\pi r} & 0 & u(\mathrm{e}^{-2\pi r}-1) \\ 0 & \mathrm{e}^{2\pi r} & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

In terms of representations this reads

$$T(u(\mathrm{e}^{-2\pi r}-1))\Delta_W^{\mathrm{i}r}.$$

Applying this to a vector of the form $A\Omega$ then $\Delta_W^{\mathrm{i}r}A\Omega, A\in\mathcal{M}(W)$ has an analytic continuation into the strip $S(-\frac{1}{2},0)$. Since the translations $T(t)$ can be continued into the upper complex half-plane, we see that $T(u(\mathrm{e}^{-2\pi r}-1))$ can be continued into $-\frac{1}{2}\le \Im mt\, t\le 0$. Therefore, the product $T(u(\mathrm{e}^{-2\pi r}-1)))\Delta_W^{\mathrm{i}r}A\Omega, A\in\mathcal{M}(W)$ also has an analytic continuation into the strip $(-\frac{1}{2},0)$. Therefore, it presents the modular group of a super-algebra of $\mathcal{M}(W)$. Writing $T(u(\mathrm{e}^{-2\pi r}-1))) = T(u(\mathrm{e}^{-2\pi r}a^+))T(-ua^+)$ we obtain with $T(-ua^+)\mathcal{M}(W)\Omega = \mathcal{M}(W(-ua^+))\Omega$ and we see that $T(u(\mathrm{e}^{-2\pi r}))$ must be the modular group of $\mathcal{M}(W(-ua^+))$. This is a shift in the negative a^+-direction. Such a situation is known from the modular action of the global algebra in thermal states. See [19].

4. The Connes-von Neumann type of local algebras

Although this problem has been solved by Fredenhagen [8], using the result of Longo [9] about the structure of the wedge-algebra, we will show the result by different methods.

Our subject is the question of the Connes-von Neumann type of the local algebras under the assumption that the local algebras are of von Neumann type III. First we have to explain the procedure, see G.K. Pedersen [20]. Let $\xi\in\mathcal{H}, \|\xi\|=1$, then we have to determine the support projection $E^\xi_{\mathcal{M}}$ of the expectation value $(\xi, .\xi)$ of $\mathcal{M}$, i.e., the smallest projection in $\mathcal{M}$ with $(\xi, E\xi)=1$. $E^\xi_{\mathcal{M}}$ is the same as the projection onto $\mathcal{M}'\xi$. Then we must compute the modular operator for the algebra $E^\xi_{\mathcal{M}}\mathcal{M}E^\xi_{\mathcal{M}}$ and its spectrum. The invariant $S(\mathcal{M})$ is obtained by the formula

$$S(\mathcal{M}) = \bigcap \mathrm{spec}\Delta_{\mathcal{M}^\xi},$$

where ξ is arbitrary and $\Delta_{\mathcal{M}^\xi}$ is the modular operator of $E^\xi\mathcal{M}E^\xi$. To determine $S(\mathcal{M})$ we will assume that $\mathcal{M}$ is a factor. This can be done without loss of generality, since we can make an integral decomposition and afterwards re-integrate the obtained results. The first result is:

4.1. Lemma:

Let $\xi\in\mathcal{H}$ and $E^\xi\in\mathcal{M}$ be the smallest projection fulfilling $E^\xi\xi=\xi$, then ξ is also cyclic and separating for $E^\xi\mathcal{M}E^\xi$.

Proof: Since $E^\xi\mathcal{H} = \mathcal{M}'\xi$ we see that ξ is cyclic for $E^\xi\mathcal{M}'E^\xi$ in $E^\xi\mathcal{H}$. On the other hand it follows that ξ is cyclic for $E^\xi\mathcal{M}E^\xi$ since E^ξ is the smallest projection in $\mathcal{M}$ with $(\xi, E^\xi\xi) = \mathbb{1}$. q.e.d.

Next we want to compare the spectra of $\Delta_\mathcal{M}$ and $\Delta_{E^\xi\mathcal{M}E^\xi}$ under the assumption that $\mathcal{M}$ is of type III. In this situation exist partial isometries $V \in \mathcal{M}$ with $VV^* = \mathbb{1}$ and $V^*V = E^\xi$.
Let U be a unitary operator in $E^\xi\mathcal{M}E^\xi$, then $VUV^* \in \mathcal{M}$. On the other hand, if $\hat{U}$ is a unitary in $\mathcal{M}$, then $V^*\hat{U}V$ is a unitary in $E^\xi\mathcal{M}E^\xi$. This means V maps all unitaries in $E^\xi\mathcal{M}E^\xi$ onto all unitaries in $\mathcal{M}$. Since the unitaries of a von Neumann algebra generate the whole algebra linearly, we obtain

$$VE^\xi\mathcal{M}E^\xi V^* = \mathcal{M}.$$

We know that ξ is cyclic and separating for $E^\xi\mathcal{M}E^\xi$. But what is with $V^*\Omega$? Next we show:

4.2. Lemma:
The vector $V^*\Omega$ is also cyclic and separating for $E^\xi\mathcal{M}E^\xi$.

Proof: Since $\mathcal{M}\Omega$ is dense in $\mathcal{H}$ and V^* is a partial isometry it follows that $E^\xi\mathcal{M}E^\xi V^*\Omega$ is dense in $E^\xi\mathcal{H}$. Assume there exists an operator $x' \in E^\xi\mathcal{M}'E^\xi$ with $E^\xi x'V^*\Omega = 0 = V^*Vx'V^*\Omega$. Now $Vx'V^* = \hat{x}'$ is an element in $\mathcal{M}'$, and since Ω is separating for $\mathcal{M}$ we obtain $V^*\hat{x}'\Omega = 0$. This implies $\hat{x}' = 0 = Vx'V^*$. Since V is a partial isometry we get $x' = 0$. q.e.d.

Unfortunately, the two vectors ξ and $V^*\Omega$ do not coincide. This defect will be cured in the next

4.3. Lemma:
For every $\xi \in \mathcal{H}$ with E^ξ support projection of $(\xi, .\xi)$ in $\mathcal{M}$ we obtain, that the modular operator of $V^*\mathcal{M}V$ and $E^\xi\mathcal{M}E^\xi$ are the same and hence we have

$$\mathrm{spec}\Delta^\xi = \mathrm{spec}\Delta^{V^*\Omega}. \qquad (4.1),$$

where $\Delta^\xi = \Delta_{E^\xi\mathcal{M}E^\xi}$ and $\Delta^{V^*\Omega} = \Delta_{V^*\mathcal{M}V}$.

Proof: Since ξ and $V^*\Omega$ both are cyclic and separating for $E^\xi\mathcal{M}E^\xi$ we get by a result of Connes (see [20] Prop. 8.14.11.) that the algebras $(E^\xi\mathcal{M}E^\xi, \mathbb{R}, \sigma^\xi)$ and $(E^\xi\mathcal{M}E^\xi, \mathbb{R}, \sigma^{V^*\Omega})$ are outer equivalent. This means there is a unitary function u_t with $\sigma^\xi(x) = u_t\sigma^{V^*\Omega}u_t^*$. By definition of u_t (see [20] 8.14.11.) one has $u_t \to \mathbb{1}$ for $t \to 0$. Applying $\frac{1}{\mathrm{i}}\frac{d}{dt}$ to the above equation we obtain in the limit $t \to \mathbb{1}$ (u_t is differentiable by its definition.):

$$\frac{1}{\mathrm{i}}\frac{d}{dt}u_t|_{t\to 0} + \Delta^{V^*\Omega} + \frac{1}{\mathrm{i}}\frac{d}{dt}u_t^*|_{t\to 0} = \Delta^\xi.$$

Since both modular operators are positive, the sum of both derivations must be selfadjoint, and since both modular operators have the eigenvalue 0, there is no shift of the spectrum. Therefore, both modular operators coincide. q.e.d.

Collecting the results obtained so far we get:

4.4. Theorem:

Using the proof of [20] lemma 8.15.8 and let μ *be a spectral point of* $\Delta_{\mathcal{M}}$ *then exists for every* $\epsilon > 0$ *an operator* $x \in \mathcal{M}$ *and a vector* y_ϵ *with* $\|y_\epsilon\| \leq \epsilon$ *and* $\Delta_{\mathcal{M}} x\Omega = \mu x\Omega + y_\epsilon$. *Now we obtain:*

$$V^*\Delta_{\mathcal{M}}VV^*x\Omega = V^*\Delta_{\mathcal{M}}x\Omega = V^*(\mu x\xi + y_\epsilon) = \mu V^*xVV^*\Omega + V^*y_\epsilon.$$

Hence spec $\Delta_{\mathcal{M}} \subset$ *spec*$\Delta^{\xi}_{E^\xi \mathcal{M} E^\xi}$. *Let* $\mathcal{M}$ *be of von Neumann type III, then the Connes-invariant* $S(\mathcal{M})$ *coincides with the spectrum of the modular operator* $\Delta_{\mathcal{M}}$.

Proof: Since for every projection $E \in \mathcal{M}$ holds spec$\Delta_{E\mathcal{M}E} \supset$ spec$\Delta_{\mathcal{M}}$ and on the other hand one has $S(\mathcal{M}) = \cap$ spec$\Delta_{E\mathcal{M}E}$, where E runs through all projections of $\mathcal{M}$, we get the result. q.e.d.

Our next aim is to try to compare for two von Neumann algebras $\mathcal{N} \subset \mathcal{M}$ their modular operators, under the assumption that both are of von Neumann type III. We start with some known results which we take from [1]. Since $(\mathcal{N}, \Omega) \subset (\mathcal{M}, \Omega)$ and since both algewbras have the same cyclic and separating vector, we obtain $\Delta_{\mathcal{N}} \geq \Delta_{\mathcal{M}}$. This is generally known, 1 proof can be found in [21]. This implies that we can form the operator valued function (see [1]):

$$F(t) = \Delta_{\mathcal{M}}^{-\mathrm{i}t}\Delta_{\mathcal{N}}^{\mathrm{i}t}.$$

This function has an analytic continuation into the strip $S(0, \frac{1}{2}), 0 < \Im m\, t < \frac{1}{2}$. This function is continuous on the boundary and norm-bounded by 1. The operator $F(\frac{\mathrm{i}}{2})$ is unitary, i.e., $F(\frac{\mathrm{i}}{2})^* = F(\frac{\mathrm{i}}{2})^{-1}$ and one gets

$$F(\frac{\mathrm{i}}{2}) = \Delta_{\mathcal{M}}^{\frac{1}{2}}\Delta_{\mathcal{N}}^{-\frac{1}{2}}.$$

Solving for $\Delta_{\mathcal{M}}^{\frac{1}{2}}$ we obtain:

$$F(\frac{\mathrm{i}}{2})\Delta_{\mathcal{N}}^{\frac{1}{2}} = \Delta_{\mathcal{M}}^{\frac{1}{2}}. \tag{4.2}$$

Since the modular operators are selfadjoint, it can be written as:

$$\Delta_{\mathcal{N}}^{\frac{1}{2}}F^*(\frac{\mathrm{i}}{2}) = \Delta_{\mathcal{M}}^{\frac{1}{2}}. \tag{4.2a}$$

This representation of $F(\frac{\mathrm{i}}{2})$ implies both equations (4.2) and (4.2a), moreover, we

obtain : 4.5. Lemma:

Between $\Delta_{\mathcal{N}}$ *and* $\Delta_{\mathcal{M}}$ *holds the relation*

$$F(\frac{\mathrm{i}}{2})\Delta_{\mathcal{N}}F^*(\frac{\mathrm{i}}{2}) = \Delta_{\mathcal{M}}. \tag{4.3}$$

This is a trivial consequence of (4,2) and (4,2a).

Eq. (4.3) allows to compare the spectra of $\Delta_{\mathcal{N}}$ and $\Delta_{\mathcal{M}}$ with a similar method as the proof of lemma 4.5.. Now we obtain:

4.6. Theorem:

Let $\mathcal{N} \subset \mathcal{M}$, and let Ω be cyclic and separating for both algebras, then we obtain:

$$\mathrm{spec}\Delta_{\mathcal{N}} = \mathrm{spec}\Delta_{\mathcal{M}}.$$

Proof: Let μ be a point in the spectrum of $\Delta_{\mathcal{N}}$, then exists for every $\epsilon > 0$ an operator $x \in \mathcal{N}$ with $\|\Delta_{\mathcal{N}} x\Omega - \mu x\Omega\| < \epsilon$, or $\Delta_{\mathcal{N}} x\Omega = \mu x\Omega + y_\epsilon$ with $\|y_\epsilon\| < \epsilon$. Multiplying this equation with $F(\frac{\mathrm{i}}{2})$ we obtain:

$$F(\frac{\mathrm{i}}{2})\Delta_{\mathcal{N}} F^*(\frac{\mathrm{i}}{2})F(\frac{\mathrm{i}}{2})x\Omega = F(\frac{\mathrm{i}}{2})(\mu x\Omega + y_\epsilon).$$

This implies together with (4.4) the equation:

$$\Delta_{\mathcal{M}} F(\frac{\mathrm{i}}{2})x\Omega = F(\frac{\mathrm{i}}{2})(\mu x\Omega + y_\epsilon).$$

Since $\mathcal{M}\Omega$ is dense in $\mathcal{H}$ exists $\tilde{x} \in \mathcal{M}$ with $F(\frac{\mathrm{i}}{2})x\Omega = \tilde{x}\Omega$, and hence

$$\Delta_{\mathcal{M}} \tilde{x}\Omega = \mu\tilde{x}\Omega + \tilde{y}_\epsilon \tag{4.4}$$

with $\tilde{y}_\epsilon = F(\frac{\mathrm{i}}{2})y_\epsilon$.
Eq. (4.4) implies:

$$\mathrm{spec}\Delta_{\mathcal{N}} \subset \mathrm{spec}\Delta_{\mathcal{M}}.$$

Passing to the commutant gives:

$$\mathrm{spec}\Delta_{\mathcal{M}}^{-1} \subset \mathrm{spec}\Delta_{\mathcal{N}}^{-1}.$$

Both equations together give the theorem. q.e.d.

Instead of using the commutaant we can solve (4.3) for $\Delta_{\mathcal{N}}$ and obtain with the similar calculation

$$\mathrm{spec}\Delta_{\mathcal{M}} \subset \mathrm{spec}\Delta_{\mathcal{N}}.$$

Up to now we have assumed that the local algebras are of type III and it remains to show that it is fulfilled.

4.7. Lemma:

The algebra $\mathcal{M}(W)$ is of von Neumann type III.

Proof: We show the lemma by contradiction. Assume $\mathcal{M}(W)$ is semi-finite then it follows that $\Delta_W^{\mathrm{i}t}$ is inner (see [20] Prop. 8.14.13.), i.e.,$\Delta_W^{\mathrm{i}t} \subset \mathcal{M}(W)$. We know for $(a, \hat{a}) \in W$ the equation $\mathrm{Ad}\,\Delta_W^{\mathrm{i}t}T(a,\hat{a}) = T(\Lambda_2 a,\hat{a})$. Let W_b be a shifted wedge, then we know for $b \in W$ $\Delta_{W_b}^{\mathrm{i}t} = T(b)\Delta_W^{\mathrm{i}t}T(-b)$ and hence

$$\Delta_{W_b}^{\mathrm{i}t}T(a)\Delta_{W_b}^{-\mathrm{i}t} = T(b)\Delta_W^{\mathrm{i}t}T(-b)T(a)T(b)\Delta_W^{-\mathrm{i}t}T(-b) = T(\Lambda_2(t)(a,\hat{a})) = \Delta_W^{\mathrm{i}t}T(a)\Delta_W^{-\mathrm{i}t}.$$

In this formula $b \in W$ denotes a n-dimensional vector, which will be written as $(b,\hat{b})$ if necessary and consequently $\Delta_W^{-\mathrm{i}t}\Delta_{W_b}^{\mathrm{i}t}T(a) = T(a)\Delta_W^{-\mathrm{i}t}\Delta_{W_b}^{\mathrm{i}t}$. Since $\Delta_W^{\mathrm{i}t}$ is inner, we get $\Delta_W^{-\mathrm{i}t}\Delta_{W_b}^{\mathrm{i}t} \subset \mathcal{M}(W) \vee \mathcal{M}(W_b)$ and with $b \in W$ we have $\Delta_W^{-\mathrm{i}t}\Delta_{W_b}^{\mathrm{i}t} \subset \mathcal{M}(W)$. On the other hand we find:

$$\Delta_W^{-\mathrm{i}t}\Delta_{W_b}^{\mathrm{i}t} = \Delta_W^{-\mathrm{i}t}T(b)\Delta_W^{\mathrm{i}t}T(-b) = T((\Lambda_2(-t)b,\hat{b}) - (b,\hat{b})).$$

Since this is for $b \in W$, $\Delta_W^{-\mathrm{i}t}\Delta_{W_b}^{\mathrm{i}t}$ contained in $\mathcal{M}(W)$ we obtain for every $B \in \mathcal{M}'(W)$ the equation $[B, T((\Lambda_2(-t)b,\hat{b}) - (b,\hat{b}))] = 0$. Multiplying with t^{-1} and going with $t \to 0$ we get $[B, T(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} b,\hat{b})] = 0$. This means $[B, T(c,\hat{b})] = 0$ for all $c \in V_2$. Since by the spectrum condition $T(.)$ has an analytic continuation into the forward tube, we obtain $T(.) \subset \mathcal{M}(W)$. **4.8. Lemma:**

$T(\lambda a^+)$ is a $(+)$half-sided translation for $\mathcal{M}(W)$. Then $T(\lambda a^+)$ is of the form $T(\lambda a^+) =$ $\mathrm{e}^{\mathrm{i}H\lambda}$,
$H \geq 0$. Let E_0 be the projection onto the $T(\lambda a^+)$ invariant states, then on $(\mathbb{1} - E_0)\mathcal{H}$ the operators $T(\lambda a^+)$ and $\Delta^{\mathrm{i}t}$ fulfil the Weyl-relation $\Delta^{\mathrm{i}t}T(\lambda a^+) = T(\mathrm{e}^{-2\pi t}\lambda a^+)\Delta^{\mathrm{i}t}$.

Proof: From [18] Theorem 2.2. we know the following: Let F_1 be the projection onto the eigenvalues 1 of $\Delta_{\mathcal{M}(W)}$ then one has $F_1 \subset E_0$. Since by assumption, Ω is the only invariant vector, we get $F_1 = E_0$. Therefore, $\log \Delta_{\mathcal{M}(W)}$ is defined on $(\mathbb{1} - E_0)\mathcal{H}$ and this space is invariant under the translation and the modular action. Now, from the relation of $\Delta^{\mathrm{i}t}$ with H, we conclude by functional calculus the relation $\mathrm{Ad}\,\Delta^{\mathrm{i}t}H^{\mathrm{i}\lambda} = \mathrm{e}^{-2\pi t\lambda}H^{\mathrm{i}\lambda}$.q.e.d.

Since the Connes invariant of $\mathcal{M}(W)$ is composed of two parts, the exponential of the spectrum of the modular group of $\mathcal{M}(W)$ on $(\mathbb{1} - E_0)\mathcal{H}$ and the value zero on $E_0\mathcal{H}$. The first part gives $\mathbb{R}^+$. Together, we obtain the closed positive half-line. With theorem 4.4. we obtain:

4.9. theorem:

The algebra of the wedge is of Connes-von Neumann type III_1.

Next we look at the cylinder in the dimension is larger than 2. In this case exists a direction $b \perp W_2$, and a translation $T(b), b \in \mathbb{R}$ under which $\mathcal{M}({}^0Z(\lambda))$ is invariant. This is the situation studied by W. Driessler [23]. Hence:

4.10. Lemma:

The algebra $\mathcal{M}({}^0Z(\lambda))$ is of von Neumann type III.

This result can also be obtained by the method described in the proof of the next theorem.

4.11. Lemma:

The algebras $\mathcal{M}({}^0D(\lambda))$ are of type III.

Proof: Let $\mathcal{M}(W)$ be a factor and let us use the methods of lemma 4.2.. Let $E \in \mathcal{M}(D(\lambda))$, then $\mathcal{M}(D(\lambda)) \subset \mathcal{M}(W)$ implies that there exists a partial isometry V with $V^*V = E$ and $VV^* = \mathbb{1} \in \mathcal{M}(W)$. Hence V has the property $VE\mathcal{M}(D(\lambda))E = \mathcal{M}(W)V$. This means as in lemma 4.2. $\mathrm{spec}\Delta_{E\mathcal{M}(D(\lambda))E} \supset \mathrm{spec}\Delta_W = \mathbb{R}^+$. Since this holds for every projection in $\mathcal{M}(D(\lambda))$, it follows that $\mathcal{M}(D(\lambda))$ is of Connes-type III_1, and this can only be true if the algebra is of von Neumann type III.

Collecting all results, obtained so far, we have:

4.12. Theorem:

All algebras we have treated are of von Neumann-type III and of Connes-type III_1.

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