Multilocal Fermionization

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Outline

- Local nets of Fermi algebras, Modular Theory on single intervals.

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- A multilocal isomorphism between one Fermi field and two Fermi fields: multilocal symmetries.

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- A multilocal isomorphism between one Fermi field and two Fermi fields: multilocal symmetries.
- Modular Theory on disjoint intervals: modular mixing.
- Generalizations: n Fermi fields and Ramond representation.

PRELIMINARIES

Fermi fields

Description by CAR$(\mathcal{H}, \Gamma) := \overline{\{\, \psi(f) \mid f \in \mathcal{H},\ (\Gamma f)(x) = \overline{f(x)}\,\}}_{\|\cdot\|}$

- The norm is uniquely fixed by

$$\psi(f)^* = \psi(\Gamma f), \qquad \{\psi(f), \psi(g)\} = (\Gamma f, g)_{\mathcal{H}}\,\mathbf{1}; \quad \forall\, f, g \in \mathcal{H}.$$

- Each generator can be decomposed into creation and annihilation modes by means of projections $P \mid \Gamma P \Gamma = \mathbf{1} - P$.

$$\psi(f) = \psi\left(Pf + \Gamma P \Gamma f\right) = \psi(Pf) + \psi(\Gamma P \Gamma f) = \psi^+(f) + \psi^-(f).$$

- **Correlation functions** emerge by projections and Wick theorem:

$$\omega_P\left(\psi(f)\psi(g)\right) := (\Gamma f, Pg)_{\mathcal{H}}$$

- Fock space = GNS representation out of ω_P

$$\{\psi^-(f), \psi^+(g)\} = (\Gamma f, Pg)_{\mathcal{H}} \qquad \text{and} \qquad \psi^-(f)\Omega_P = 0.$$

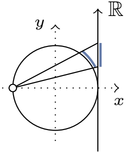
Standard notations

Fields will be equivalently expressed both as distributions

$$\{\psi(x), \psi(y)\} = \delta(x - y)$$

and as smeared operators, according to what we like most. Also, the real line $\mathbb{R}$ is mapped onto the unit circle S^1 via the Cayley map

$$x \to \frac{1 + ix}{1 - ix} = z$$



- Real Fermi field: $\mathcal{H} = \mathrm{L}^2(\mathrm{S}^1)$. No internal symmetries occur.
- Complex Fermi field: $\mathcal{H} = \mathrm{L}^2(\mathrm{S}^1) \oplus \mathrm{L}^2(\mathrm{S}^1) = \mathrm{L}^2(\mathrm{S}^1) \otimes \mathbb{C}^2$. Internal $\mathrm{U}(1)$ gauge symmetry and currents.

Representations

Two positive energy representations:

Vacuum state: $$\omega_0(\psi(z)\psi(w)) = \lim_{|\frac{z}{w}|\to 1} \frac{1}{z-w}$$

Ramond state: $$\omega_{\mathsf{R}}(\psi(z)\psi(w)) = \lim_{|\frac{z}{w}|\to 1} \frac{1}{2\sqrt{zw}}\frac{z+w}{z-w}$$

They give rise, via GNS, to:

$$\pi_0(\psi(z)) = \sum_{n\in\mathbb{Z}+\frac{1}{2}} \psi_{0,n}\ z^{-n-1/2}$$ periodic on S^1

$$\pi_{\mathsf{R}}(\psi(z)) = \sum_{n\in\mathbb{Z}} \psi_{\mathsf{R},n}\ z^{-n-1/2}$$ anti-periodic on S^1

Modular Theory on single interval: ω_0

Choose $\mathcal{A}(O)$ as vN algebra and the vacuum state as Ω. Then σ^t is the one-parameter subgroup $\subset$ Möb preserving the interval:

$$\sigma^t\left(\psi(z)\right) = \sqrt{\delta'_{-2\pi t}(z)}\,\psi\left(\delta_{-2\pi t}(z)\right)$$

In two intervals $\mathrm{E} = \mathsf{I} \cup -\mathsf{I}$ the action **cannot** be entirely geometric in each interval anymore because of Takesaki's theorem!

If $\sigma^t_{\mathrm{E}}(\mathcal{A}(\mathsf{I})) \subset \mathcal{A}(\mathsf{I}) \implies \exists\, \mathcal{E} \colon \mathcal{A}(\mathrm{E}) \to \mathcal{A}(\mathsf{I})$ such that:

$$\mathcal{E}(x) = x,\ x \in \mathcal{A}(\mathsf{I}); \qquad \mathcal{E}(x)\,\Omega = P_{\mathsf{I}}\,x\,\Omega,\ x \in \mathcal{A}(\mathrm{E})$$

where P_{I} projects onto $\mathcal{H}_{\mathsf{I}} = \{\, x\,\Omega \mid x \in \mathcal{A}(\mathsf{I})\,\}$. Because of Reeh-Schlieder Ω is also cyclic for $\mathcal{A}(\mathsf{I})$, hence $\mathcal{H}_{\mathsf{I}}$ is dense in $\mathcal{H}$. It follows that $P_{\mathsf{I}} = \mathbf{1}$ and

$$\mathcal{E}(x)\,\Omega = x\,\Omega \quad \forall x \in \mathcal{A}(\mathrm{E}); \implies \mathcal{E}(x) = x \quad \forall x \in \mathcal{A}(\mathrm{E})$$

thus this implies $\mathcal{A}(\mathrm{E}) = \mathcal{A}(\mathsf{I})$, which is not the case at hand.

A MULTILOCAL ISOMORPHISM

The isomorphism β

Proposition:

$$\beta\colon \mathsf{CAR}^{(\mathbb{C})}(\mathrm{S}^1) \to \mathsf{CAR}^{(\mathbb{R})}(\mathrm{S}^1)$$

$$\phi(z^2) \to \frac{1}{2}\left(\psi(z) + \psi(-z)\right); \qquad \phi^*(z^2) \to \frac{1}{2z}\left(\psi(z) - \psi(-z)\right)$$

is an isomorphism preserving the vacuum state.

Local proof: compute the 2-point function:

$$\omega_0 \circ \beta\left(\phi^*(z^2)\phi(w^2)\right) = \omega_0\left(\phi^*(z^2)\phi(w^2)\right) = \frac{1}{z^2 - w^2}$$

Global proof: relabel Fourier modes:

$$\phi_n \to \beta(\phi_n) = \psi_{2n+\frac{1}{2}}, \qquad \phi_n^* \to \beta(\phi_n^*) = \psi_{2n-\frac{1}{2}}, \qquad n \in \mathbb{Z} + \frac{1}{2}.$$

Change of localization

Notice: the real Fermi field ψ is located at two antipodal points $\pm z \in \mathrm{S}^1$ while the complex Fermi field ϕ is located at $z^2 \in \mathrm{S}^1$.

Observables: since β preserves the vacuum state it extends to Wick products. Observables may be embedded to give rise to "multilocal" symmetries.

- Stress-energy tensor generates diffeomorphisms:

$$\mathrm{e}^{iT(f)}\,\phi(z)\,\mathrm{e}^{-iT(f)} = \gamma'(z)\,\phi(\gamma_f(z)).$$

- Currents generate gauge transformations:

$$\mathrm{e}^{ij(f)}\,\phi(z)\,\mathrm{e}^{-ij(f)} = \mathrm{e}^{-if(z)}\,\phi(z).$$

Multilocal symmetries

"Multilocal Fermionization"

$$\beta\left(j(z^2)\right) = \frac{1}{2z} :\psi(z)\psi(-z):$$

$$\beta\left(W(f)\right)\psi(z)\,\beta\left(W(f)^*\right) = \cos f(z^2)\,\psi(z) + \sin f(z^2)\,\psi(-z).$$

"Multilocal Diffeomorphisms"

$$\beta\left(T^{c=1}(z^2)\right) = -\frac{1}{8\pi z^2}\,\beta\left(j(z)\right) + \frac{1}{4z^2}\left(T^{c=\frac{1}{2}}(z) + T^{c=\frac{1}{2}}(-z)\right)$$

Again, under the action of β, we have a mixing of $\psi(z)$ and $\psi(-z)$ due to the first contribution, on top of a geometric flow due to the second one.

MULTI-INTERVALS MODULAR THEORY

Modular theory on disjoint intervals: ω_0

Since $\omega_0 \circ \beta = \omega_0 \otimes \omega_0$ then β intertwines the respective modular groups

$$\sigma_t^{(\mathbb{R})} \circ \beta = \beta \circ \sigma_t^{(\mathbb{C})}.$$

This allows to independently compute the modular group for 2-intervals, provided you know the modular group for 1-interval.

$$\sigma^t_{\mathsf{I}\cup -\mathsf{I}} = \beta \circ \sigma^t_{\mathsf{I}^2} \circ \beta^{-1}; \qquad \sigma^t_{\mathsf{I}^2} = \mathrm{Ad}\,\mathrm{U}\left(\delta_{-2\pi t}\right)$$

The novelty is that, due to the **non-locality of β**, the modular group $\sigma^t_{\mathsf{I}\cup -\mathsf{I}}$ **mixes** the components in 2-intervals (remember Takesaki's theorem):

$$\sigma^t_{\mathsf{I}\cup -\mathsf{I}} \begin{pmatrix} \psi(z) \\ \psi(-z) \end{pmatrix} = O(t,z) \begin{pmatrix} \psi\left(\delta^{(2)}_{-2\pi t}(z)\right) \\ \psi\left(\delta^{(2)}_{-2\pi t}(-z)\right) \end{pmatrix}$$

Another way to look at it

The modular group in 1-interval is given by dilations $\sigma^t_{\mathsf{I}^2} = \mathrm{Ad}\,\mathrm{U}\left(\delta_{-2\pi t}\right)$. They arise as generated by the local SET $T^{c=1}(f)$.

Its embedding under the multilocal map β delocalizes the components, therefore the adjoint action of $\beta\left(T^{c=1}(f)\right)$ happens to delocalize fields.

$$[T^{c=1}(f), \psi(z^2)] \to \sigma^t_{\mathsf{I}^2}\left(\psi(z^2)\right) \qquad \text{"local" dilations}$$

$$[\beta\left(T^{c=1}(f)\right), \psi(z)] \to \sigma^t_{\mathsf{I}\cup -\mathsf{I}}\left(\psi(z)\right) \qquad \text{"multilocal" dilations}$$

namely, the computation of the LHS allows to derive the RHS.

Thanks to...

The formula for the modular mixing has been computed for the first time by Casini & Huerta[1]

$$\sigma^t_{I\cup -I}\begin{pmatrix}\psi(z)\\ \psi(-z)\end{pmatrix} = \begin{pmatrix}c_1(z,t) & c_2(z,t)\\ -c_2(z,t) & c_1(z,t)\end{pmatrix}\begin{pmatrix}\psi\left(\delta^{(2)}_{-2\pi t}(z)\right)\\ \psi\left(\delta^{(2)}_{-2\pi t}(-z)\right)\end{pmatrix}$$

where $c_1(z,t)^2 + c_2(z,t)^2 = 1$ and $\delta^{(2)}_{-2\pi t}(z) = \sqrt{\delta^{(1)}_{-2\pi t}(z^2)}$.

Roughly speaking look for an isomorphism $\beta\colon \psi \to B\,\psi$ such that

$$B(z)\,O(z,t)\,B(z)^{-1} = \text{diagonal}$$

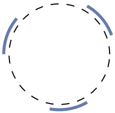
Thus β **diagonalizes** the modular mixing.

[1] H. Casini, M. Huerta: Class. Quant. Grav. **26** (2009) 185005 [arXiv:0903.5284 [hep-th]].

MANY INTERVALS AND RAMOND REPRESENTATION

n-interval case

Picture: symmetric n-interval on the circle.



Write n real fields as n complex fields being mutually conjugated each other $\left(\phi^{(k)}\right)^* = \phi^{(n+1-k)}(x)$. Let $\omega \mid \omega^n = 1$, the following

$$\beta \colon \phi^{(k)}(z^n) \to \frac{z^{1-k}}{n} \sum_{j=0}^{n-1} \omega^{(1-k)j}\, \psi(\omega^j z) \qquad \left(z \in \mathrm{S}^1, k = 1, \ldots, n\right)$$

satisfies $\omega_0 \circ \beta = \underbrace{\omega_0 \otimes \ldots \otimes \omega_0}_{\longleftarrow\, n\, \longrightarrow}$ and it intertwines

$$\sigma^t_{\bigcup_j \omega^j \mathsf{I}} = \beta \circ \sigma^t_{\mathsf{I}^n} \circ \beta^{-1}$$

reproducing Casini & Huerta, as supposed.

Ramond representation

$\mathsf{CAR}(\mathbb{R})$ possesses another positive energy representation, i.e. Ramond representation induced via GNS from

$$\omega_{\mathsf{R}}(\psi(x)\psi(y)) = \lim_{\epsilon\to 0^+} \frac{1+xy}{\sqrt{1+x^2}\sqrt{1+y^2}} \frac{-i}{x-y-i\epsilon} \,.$$

Along the same lines as before one can prove the existence of

$$\beta_{\mathsf{R}} \colon \mathsf{CAR}_{\mathsf{R}}(\mathrm{S}^1) \otimes^t \mathsf{CAR}_0(\mathrm{S}^1) \to \mathsf{CAR}_{\mathsf{R}}(\mathrm{S}^1)$$

$$\psi_{\mathsf{R}}(z^2) \otimes^t \mathbf{1}_0 \to \frac{1}{2}\left(\psi_{\mathsf{R}}(z) + \psi_{\mathsf{R}}(-z)\right); \qquad \mathbf{1}_{\mathsf{R}} \otimes^t \psi_0(z^2) \to \frac{1}{2z}\left(\psi_{\mathsf{R}}(z) - \psi_{\mathsf{R}}(-z)\right)$$

$$\omega_{\mathsf{R}} \circ \beta_{\mathsf{R}} = \omega_{\mathsf{R}} \otimes \omega_0$$

Ramond representation

From the previous it follows that

$$\sigma^t_{\mathsf{R},\,\mathsf{I}\cup-\mathsf{I}} = \beta_\mathsf{R} \circ \left(\sigma^t_{\mathsf{R},\,\mathsf{I}^2} \otimes \sigma^t_{0,\,\mathsf{I}^2}\right) \circ \beta_\mathsf{R}^{-1}$$

with $\sigma_{\mathsf{R},\,\mathsf{I}^2}$ still some unknown geometric (?) action. Apply to the RHS of the above map, first to $\lambda_\mathsf{R}(z) = \beta_\mathsf{R}(\mathbf{1}_\mathsf{R} \otimes^t \psi_0(z^2))$

$$\begin{aligned}
\sigma^t_{\mathsf{R},\,\mathsf{I}\cup-\mathsf{I}}\left(\lambda_\mathsf{R}(z)\right) &= \beta_\mathsf{R} \circ \left(\sigma^t_{\mathsf{R},\,\mathsf{I}^2} \otimes \sigma^t_{0,\,\mathsf{I}^2}\right) \circ \beta_\mathsf{R}^{-1}\left(\lambda_\mathsf{R}(z)\right) \\
&= \beta_\mathsf{R} \circ \left(\sigma^t_{\mathsf{R},\,\mathsf{I}^2} \otimes \sigma^t_{0,\,\mathsf{I}^2}\right)\left(\mathbf{1}_\mathsf{R} \otimes \psi_0(z^2)\right) \\
&= \beta_\mathsf{R}\left(\mathbf{1}_\mathsf{R} \otimes \sqrt{\delta'^{(\mathsf{I}^2)}_{-2\pi t}(z^2)}\;\psi\left(\delta^{(\mathsf{I}^2)}_{-2\pi t}(z^2)\right)\right) \\
&= \sqrt{\delta'^{(\mathsf{I}^2)}_{-2\pi t}(z^2)}\;\lambda_\mathsf{R}\left(\sqrt{\delta^{(\mathsf{I}^2)}_{-2\pi t}(z^2)}\right) \\
&= \sqrt{\delta'^{(\mathsf{I}^2)}_{-2\pi t}(z^2)}\;\lambda_\mathsf{R}\left(\delta^{(\mathsf{I}\cup-\mathsf{I})}_{-2\pi t}(z)\right)
\end{aligned}$$

hence **geometric 2-intervals** action w.r.t. 2-dilations on $\lambda_\mathsf{R}(z)$.

Ramond representation

On the other hand little can be said if you apply the previous to $\mu_{\mathsf{R}}(z)$, i.e. $\mu_{\mathsf{R}}(z) = \beta_{\mathsf{R}}(\psi_{\mathsf{R}}(z^2) \otimes^t \mathbf{1}_0)$

$$\begin{aligned}\sigma^t_{\mathsf{R},\, \mathsf{I}\cup -\mathsf{I}}\left(\mu_{\mathsf{R}}(z)\right) &= \beta_{\mathsf{R}} \circ \left(\sigma^t_{\mathsf{R},\, \mathsf{I}^2} \otimes \sigma^t_{0,\, \mathsf{I}^2}\right) \circ \beta_{\mathsf{R}}^{-1}\left(\mu_{\mathsf{R}}(z)\right) \\ &= \beta_{\mathsf{R}} \circ \left(\sigma^t_{\mathsf{R},\, \mathsf{I}^2} \otimes \sigma^t_{0,\, \mathsf{I}^2}\right)\left(\psi_{\mathsf{R}}(z^2) \otimes \mathbf{1}_0\right) \\ &= \beta_{\mathsf{R}} \circ \left(\sigma^t_{\mathsf{R},\, \mathsf{I}^2}\left(\psi_{\mathsf{R}}(z^2)\right) \otimes \mathbf{1}_0\right)\end{aligned}$$

now what about $\sigma^t_{\mathsf{R},\, \mathsf{I}^2}\left(\psi_{\mathsf{R}}(z^2)\right)$? Invariance arguments show that the action **cannot** be geometric inside 1 interval!

2-intervals: geometric on $\lambda_{\mathsf{R}}(z)$; non-geometric on $\mu_{\mathsf{R}}(z)$;

1-interval: non-geometric.

BRIEF SUMMARY

Brief summary

Key ideas:

- $\exists$ an isomorphism $\beta\colon \mathsf{CAR}^n(\mathrm{S}^1) \to \mathsf{CAR}(\mathrm{S}^1),\ \omega_0 \circ \beta = \omega_0$.
- β is "multilocal", i.e. fields in $z \in \mathrm{S}^1$ are mapped into fields at $\omega^j z$.
- Observables (currents and SET) are embedded via β. New multilocal symmetries arise.
- β gives an underlying explanation for the Modular Theory of free Fermi fields on disjoint intervals: namely the modular mixing is directly traced back to the non-local isomorphism.
- Attempts to derive properties of the Ramond representation.