Thermal states of deformed quantum field theories

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Motivation from physics

- High energy physics: Focus often on aspects of interaction tested by scattering experiments (few particles $\Rightarrow$ S-matrix)
- Thermal behavior of system described by given theory also of practical and conceptual interest:
  - Description of early universe, heavy ion collisions (quark-gluon plasma) etc.
  - Thermal equilibrium states "preferred by physics" (return to equilibrium).
  - Requirement of decent thermodynamical behavior: Selection criterion for theories.
- Deformed QFTs: Examples of interacting theories
  $\Rightarrow$ non-pert. relativistic thermodyn. beyond the free fields

Motivation from the deformation programme

Obtaining QFTs by algebraic methods

- Starting with [Lechner 06]: Construction of interacting quantum fields theories by algebraic methods via (auxiliary) wedge local nets of algebras.
  In 2d: Can proceed to local algebras.
- Wedge algebras obtained via "deformation" from given (usually free) QFT.
- Direct interpretation for (special) wedge algebras in non-commutative models.
- In more than 2 dimensions: Step to local algebras not (yet?) useful.

Motivation from the deformation programme

Deformations and the spectrum condition

- Family of deformations leading to wedge-local theories still limited (e.g. for massive theories: no momentum transfer in scattering).
- Important for wedge locality in these constructions: Spectrum condition.
- Thermal representation would be example where wedge-locality holds in situation without spectrum condition.

Undeformed object: Free scalar field

Aim: Keep algebra as small as possible (still large!)
Work with *-algebra generated by deformed fields $\phi_{R,Q}$ obtained by deforming free field (in vacuum representation).

- Undeformed field: Free scalar field ϕ of mass m on 3+1 dimensional Minkowski spacetime.
- Acting on Fock-space $\mathcal{H} = \mathbb{C} \oplus \bigoplus_{n=1}^{\infty} \mathcal{H}_n$, $\mathcal{H}_1 = L^2(\mathbb{R}^4, d\mu_m)$.
- $\varphi = a^\dagger(Ef) + a(E\overline{f})$, $(Ef)(\mathbf{p}) := \hat{f}(\sqrt{\mathbf{p}^2 + m^2}, \mathbf{p})$
- Unitary, positive energy repr. U of Poincaré group $\mathcal{P}^\uparrow$ on $\mathcal{H}$ (obtained as $\Gamma(U_1)$ from representation on $\mathcal{H}_1$).
- Commutator: $[\phi(f), \phi(g)] = C(f, g) \cdot 1$; Fourier-transform of (distributional) kernel of C: $\hat{C}$.

Input of deformations

- Class of deformations: Multiplicative deformation of free theories as described in [Lechner 11].
- Deformations parametrized by
  - Bounded analytic function φ on upper half-plane satisfying $\overline{\varphi(t)} = \varphi(t)^{-1} = \varphi(-t)$ for $t \in \mathbb{R}$.
  - Enters deformation via $R \in L^\infty(\mathbb{R})$ s.t. $R(t)^2 = \varphi(t)$, $\overline{R(t)} = R(t)^{-1} = R(-t)$.
  - Special case: $R(t) = e^{it}$; choice for some parts of talk (simplifications in formulas).
  - Lorentz-antisymmetric deformation matrix

$$Q_0 = \begin{pmatrix} 0 & \kappa & 0 & 0 \\ \kappa & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda \\ 0 & 0 & -\lambda & 0 \end{pmatrix}$$

Deformed fields

- Using R define unitaries $T_R(x)$ by

$$[T_R(x)\Psi]_n(p_1,\ldots,p_n) := \prod_{k=1}^{n} R(x\cdot p_k)\Psi_n(p_1,\ldots,p_n)$$

- Using them define deformed field by

$$a_{R,Q}(p) := a(p)\,T_R(Qp) \qquad a^\dagger_{R,Q}(p) := a_{R,Q}(p)^\dagger$$

$$\phi_{R,Q}(f) = a^\dagger_{R,Q}(Ef) + a_{R,Q}(E\overline{f})$$

Properties of generators

- We have

$$U(\Lambda, a)\phi_{R,Q}(f)U(\Lambda, a)^{-1} = \phi_{R,\Lambda Q\Lambda^{-1}}(f(\Lambda^{-1}\cdot - a))$$

- For the product of two (Fourier transformed) fields (formally):

$$\hat{\phi}_{R,Q}(p)\hat{\phi}_{R,Q'}(p') - R(p\cdot Qp')R(p\cdot Q'p')\hat{\phi}_{R,Q'}(p')\hat{\phi}_{R,Q}(p) = T_R(Qp)T_R(-Q'p)\hat{C}(p,p')\cdot 1 \qquad (1)$$

- $\phi_{R,Q_0}(f)$ with $\operatorname{supp} f \subset W_R$ is localized in the right-wedge $W_R = \{x \in \mathbb{R}^4 \mid x^1 > |x^0|\}$.

The *-algebra

- Localization and covariance properties of $\phi_{R,Q}$ strongly suggest:

$$\mathcal{A}(W = \Lambda W_R + x) := \{A \mid A \text{ pol. in } \phi_{\Lambda Q_0 \Lambda^{-1}}(f), \operatorname{supp} f \subset W\}$$

- (Some) known results
  - This gives an isotonous, Poincaré covariant wedge-local net of $* -$ *algebras* [Lechner 11].
  - Different choices of roots R of the given function φ result in equivalent nets. [Lechner/Tanimoto/S 12].

Reduction of problem and crossed products

Now: Determine KMS-states ω_β on $\mathcal{A}$, $R(t) = e^{it}$, $\phi_Q := \phi_{R,Q}$

- Can reduce monomial $A = \hat{\phi}_{Q_1}(p_1)\cdots\hat{\phi}_{Q_n}(p_n)$ to

$$\prod_{1\leq l<r\leq n} e^{ip_l\cdot Qp_r}\hat{\phi}(p_1)\cdots\hat{\phi}(p_n)U(Q_1p_1+\ldots+Q_np_n)$$

  $\Rightarrow$ State ω determined by $\omega_x(B) := \omega(BU(x))$, B a monomial in free (Fourier-transformed) fields.

- ω_x has to satisfy *twisted* KMS condition [Buchholz/Longo 99]:

$$\omega_x(A\alpha_{(t+i\beta)e}B) = \omega_x((\alpha_{te}B)\alpha_x A)$$

  $\alpha_x(A) := U(x)AU(x)^{-1}$, e a timelike unit vector.

- In C^*-algebraic setting: Can realize deformed algebra as multiplier algebra of crossed product $\mathcal{A}\rtimes_\alpha \mathbb{R}^4$, reduction to functionals ω_x and twisted KMS-condition follows in general.

Explicit form of KMS functionals

- Using (1) and the same technique as for free fields, a recursion relation for ω_x can be obtained.
- From this: All odd n-point functions vanish.
- Even n-point functions given by

$$\omega_\beta(\hat{\phi}_{Q_1}(p_1)\cdots\hat{\phi}_{Q_n}(p_{2n}) = w(-x) \prod_{1\leq l<r\leq n} e^{ip_l \cdot Qp_r} \times \dots$$

$$\dots \times \sum_{\text{Pair.}(\mathbf{l},\mathbf{r})} \prod_{k=1}^{n} \frac{\hat{C}(p_{l_k}, p_{r_k})}{1-\exp\left(p_{l_k}(\beta e + ix)\right)}$$

$x := \sum_{j=1}^n Q_j p_j$.

KMS functionals: Remarks

- Appearance of function w makes KMS functionals highly non-unique.
- They are invariant under translations and rotation
- With $w(0) = 1$ normalization and reality ($\omega_\beta(A^*) = \overline{\omega(A)}$) also hold.

But what about positivity?

Subalgebras with fixed deformation matrix

- Polynomials in deformed fields with same deformation matrix Q form subalgebra $\mathcal{A}_Q$.
- $\mathcal{A}_Q$ invariant under translations.
  $\Rightarrow$ Restr. of ω_β to $\mathcal{A}_Q$ gives KMS state on $\mathcal{A}_Q$.
- Function w and additional term in Bose-factor disappear ($x = 0$).
- Remaining functionals still differ from undeformed case (by phase factors) but positivity can be shown.

A special functional

- "Traditional" method to obtain KMS states:
  Put system in box $\Lambda \Rightarrow$ Gibbs states $\omega_{\beta,\Lambda} = \frac{1}{Z}\mathrm{Tr}\left(e^{-\beta H_\Lambda}\ \cdot\ \right)$
  $\Rightarrow$ Obtain KMS states as $\lim_{\Lambda\to\mathbb{R}^3}\omega_{\beta,\Lambda}$.
- $U \Rightarrow H$ identical for deformed and undeformed theory
  $\Rightarrow$ Gibbs states also agree $\Rightarrow$ State as in textbook statistical mechanics of free Bose gas, but consider expectation values of different operators A_Q.
- After limit: KMS functional w. non-continuous function w:

$$w(t) = \begin{cases} 1 & t = 0 \\ 0 & \text{else} \end{cases}$$

- For this functional: Many contractions vanish; only obviously positive terms remain $\Rightarrow$ KMS *state*.
- State leads to representation on non-separable Hilbert-space.

Some positivity from continuity of the deformation

- Know positivity of KMS functionals for undeformed theory
- For $w = 1$: n-point functions of deformed states depend continuously on Q_0.
- For given polynomial A_{Q_0} in deformed field (i.e. test functions and wedges to which fields belong fixed):

$$\omega(A^*_{Q_0} A_{Q_0}) \geq 0$$

for $\|Q_0\| < \delta_A$.

- However: δ_A depends on choice of polynomial, minimum over all polynomials in algebra may be zero.

Numerical search for counterexamples

- Explicit knowledge of n-point functions makes automated search for negative expectation values possible.
- Implemented functions to calculate $\omega_\beta(A^*A)$ for arbitrary field polynomial; automated search for counterexamples by checking positivity of randomly generated polynomials A.
- So far: Checks up to 8-point functions (test-functions: polynomials $\times$ shifted, scaled Gaussian) No negative values encountered so far!
- Even stronger positivity property seems to be true: All discrete approximands to expectation values (integrals -> sums) positive.

Summary

- Thermal states provide additional insight into models obtained by deformations.
- KMS-functionals for *-algebras generated by deformed fields can be explicitly calculated.
- On subalgebras with fixed deformation matrix these correspond to states of the undeformed algebra.
- On the whole algebra their structure is more complicated + uniqueness breaks down (function w).
- Positivity of the functionals is hard to decide; there is however a somewhat singular state obtained by approximations from finite volume.

TODO

- Put suitable topology on *-algebra, make some of the calculations more precise.
- Determine which of the KMS functionals are positive.