Coherent States and the Semi-Classical Einstein Equation

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The main motivation is solving the semi-classical Einstein equation:

$$8\pi G\,\omega(:T_{\mu\nu}:) = G_{\mu\nu}$$

We try to find a proper state ω whose energy-momentum tensor matches the right hand side of the semi-classical Einstein equation.

Coherent states can be the states we are looking for, since these states have the most classical behavior among the quantum states.

One usually constructs coherent states with respect to a ground state, but on general curved spacetimes there is no ground state. Therefore one has to use a generalised ground state; here we consider states of low energy as generalised ground states.

The Klein-Gordon equation in a general curved spacetime is:

$$(-\Box + m^2 + \xi R)\phi = 0$$

where ξ is a coupling constant and R is the Ricci scalar.
The algebra $\mathscr{A}$ of the Klein-Gordon field is generated by the unit element I and the smeared fields $\phi(g)$, formally defined as:

$$\phi(g) = \int dx\, \phi(x) g(x) \tag{1}$$

where $g(x)$ is a suitable test function. Moreover, the Klein-Gordon equation and the canonical commutation relations are included in $\mathscr{A}$ as suitable algebraic relations.

Coherent states on curved spacetime

Coherent states ω_f on curved spacetime are defined with respect to generalised ground states and characterised by a classical solution f of the Klein-Gordon equation.

Definition

We define an automorphism $\alpha_f : \mathscr{A} \to \mathscr{A}$ as follows:

$$\alpha_f[\phi(g)] := \int \phi(x)g(x)dx + \int f(x)g(x)dx,$$

where $g(x)$ is a test function and $f(x)$ is a solution of Klein-Gordon equation. Then if ω is the generalised ground state on $\mathscr{A}$, $\omega \circ \alpha_f$ is another state, which is called a *coherent state*.

The Semi- Classical Einstein equation and Energy-Momentum Tensor of Coherent states

For studying the backreaction of quantum fields on the background spacetime we use the semi-classical Einstein equation:

$$8\pi G\,\omega(:T_{\mu\nu}:) = G_{\mu\nu}$$

we use the energy-momentum tensor of coherent states as an ansatz in semi-classical Einstein equation. The energy-momentum tensor of a coherent state ω_f looks like:

$$\omega_f(:T_{\mu\nu}:) = \omega(:T_{\mu\nu}:) + T_{\mu\nu}(f)$$

where ω is a generalised ground state and f is a solution of Klein-Gordon equation.

In the case of a homogeneous and isotropic spacetime, it suffices to consider the 00-component of this equation, i.e. on the left hand side the energy density:

$$\omega_f(:\rho:) := \omega_f(:T_{00}:) = \omega(:\rho:) + \rho(f)$$

In my work, on a given spacetime, we compute the energy density of a general coherent state and try to find an f such that the semi-classical Einstein equation is satisfied in the related coherent state.
This general idea will be applied in three cases:

- Calculating the expectation value of the ground state energy density in the 3D torus spacetime – the Casimir effect – and using in order to solve the semi-classical Einstein equation in this spacetime
- Considering states of low energy and their corresponding coherent states in de Sitter spacetime and trying to solve the semi-classical Einstein equation by means of them
- Solving the semi-classical Einstein equation in general Robertson-Walker spacetimes by means of coherent states with respect to states of low energy under the assumption that the energy density of the latter is negligible

We want to calculate the expectation value of the energy density in the 4-dimensional flat spacetime $M = \mathbb{R} \times T^3$, where T^3 is the 3-torus with radius R.

the Klein-Gordon field on the torus spacetime

The Klein-Gordon field on 3-dimensional torus spacetime in the representation induced by the vacuum state k is given by:

$$\phi(t,\mathbf{x}) = (2\pi R)^{-\frac{3}{2}} \sum_{\mathbf{n}\in\mathbb{Z}^3} (2\omega_{\mathbf{n}})^{-\frac{1}{2}} [a_{\mathbf{n}} e^{-i[\omega_{\mathbf{n}} t - \frac{\mathbf{n}\cdot\mathbf{x}}{R}]} + a^{\dagger}_{\mathbf{n}} e^{i[\omega_{\mathbf{n}} t - \frac{\mathbf{n}\cdot\mathbf{x}}{R}]}]$$

where $\omega_{\mathbf{n}} = (\frac{\mathbf{n}^2}{R^2} + m^2)^{\frac{1}{2}}$ and we consider $\frac{\mathbf{n}}{R}$ as the momentum in torus spacetime which can possess only discrete values because of the boundary conditions.

the expectation value of normal ordered energy density is obtained w.r.t. vacuum state ω on Minkowski spacetime:

$$k(:\rho(z):) = k(\rho(z)) - \omega(\rho(z))$$

the energy density for the massive Klein-Gordon field at the point z is given by:

$$\rho(z) = \frac{1}{2}\left[\left(\frac{\partial\phi}{\partial t}\right)^2 |_z + \left(\vec{\nabla}\phi\right)^2 |_z + m^2\phi^2 |_z\right]$$

we use the point-splitting prescription and and translation invariance to represent the expectation value of the normal ordered energy density by:

$$k(:\rho(z):) = \lim_{x\to 0}[-\frac{1}{2}(\partial_t^2 + \Delta - m^2)[k(\phi(x)\phi(0)) - \omega(\phi(x)\phi(0))]] \tag{2}$$

We reformulate the two-point function of k with the help of Poisson resummation.

Poisson Resummation

The Poisson resummation is given by:

$$\sum_{n \in \mathbb{Z}^3} f(n) = \sum_{m \in \mathbb{Z}^3} \int f(n) e^{-im2\pi n} dn$$

where f is a function of a integer number n.

Then the two-point function looks like:

$$k(\phi(x)\phi(0)) = \sum_{\mathbf{n} \in \mathbb{Z}^3} \omega(\phi(x + 2\pi R n)\phi(0))$$

In the massless case, we then obtain:

$$k(:\rho:) = \sum_{\mathbf{n}\in\mathbb{Z}^3\setminus\{0\}} -\frac{2}{(2\pi)^4 R^4 \mathbf{n}^4}$$

this sum is convergent and we call it Γ_1. Now we have the energy density of the ground state, then we should find a homogeneous and isotropic solution f of the Klein-Gordon equation whose energy density satisfies the semi-classical Einstein equation.

Einstein Tensor

On the flat torus spacetime we have no curvature, therefore:

$$R_{\mu\nu\rho\sigma} = 0, \qquad G_{\mu\nu} = 0$$

then the right hand side of semi-classical Einstein equation is zero.

For a homogeneous and isotropic solution which only depends on t, we have:

$$\rho(f) + k(:\rho:) = \frac{1}{2}(\dot{f}(t))^2 + k(:\rho:) = 0$$

the solution for function f is:

$$f(t) = (-2\Gamma_1)^{\frac{1}{2}} t + c$$

For the massive case, the normal ordered expectation value of energy density is obtained analogously, but from the massive two-point functions.

Again we use the Poisson resummation for the massive two-point function of the ground state in 3D torus spacetime; then the expectation value of the normal ordered energy density is given by:

$$k(:\rho:) = \lim_{\mathbf{x}\to 0}[-\frac{1}{2}(\partial_t^2 + \Delta - m^2) \sum_{\mathbf{n}\in\mathbb{Z}^3\setminus\{0\}} \omega(\phi(\mathbf{x} + 2\pi R\mathbf{n})\phi(0))]$$

The massive two-point function

the two-point function of the vacuum state in the massive case is:

$$\omega(\phi(x)\phi(0)) = \lim_{\epsilon\downarrow 0} \frac{4m}{(4\pi)^2(\sigma_\epsilon(x,0))^{\frac{1}{2}}} K_1(m(\sigma_\epsilon(x,0))^{\frac{1}{2}})$$

where $\sigma_\epsilon(x,0)$ is the half squared geodesic distance and K_1 is a modified Bessel function of order one.

Here we restrict to asymptotic case for which, $mR \gg 1$. The modified Bessel function $K_1(x')$ behaves asymptotically as follows:

$$K_1(x') \simeq e^{-x'} \left[\sqrt{\frac{\pi}{2}} \sqrt{\frac{1}{x'}} + O\left(\left(\frac{1}{x'} \right)^{3/2} \right) \right]$$

After taking the derivatives and impose the limit, the normal ordered expectation value of energy density looks like:

$$k(:\rho:) = -\frac{1}{2} \sum_{\mathbf{n}\in\mathbb{Z}^3\setminus\{0\}} \left[\frac{3\sqrt{m\pi/2}}{4\pi^2} \frac{e^{-m((2\pi n_1 R)^2+(2\pi n_2 R)^2+(2\pi n_3 R)^2)^{\frac{1}{2}}}}{((2\pi n_1 R)^2+(2\pi n_2 R)^2+(2\pi n_3 R)^2)^{\frac{7}{4}}} + \frac{m\sqrt{m\pi/2}}{2\pi^2} \frac{e^{-m((2\pi n_1 R)^2+(2\pi n_2 R)^2+(2\pi n_3 R)^2)^{\frac{1}{2}}}}{((2\pi n_1 R)^2+(2\pi n_2 R)^2+(2\pi n_3 R)^2)^{\frac{5}{4}}} \right]$$

the result of this sum is convergent and we call it Γ_2. Then the semi-classical Einstein equation, after putting the last result in it, is given by:

$$\frac{1}{2}(\dot{f}^2 + m^2 f^2) + \Gamma_2 = \frac{G_{\mu\nu}}{8\pi G}$$

where the right hand side is equal to zero; then the proper homogeneous solution f of Klein-Gordon equation is:

$$f = \frac{(2|\Gamma_2|)^{\frac{1}{2}}}{m} \sin(mt + c)$$

States of Low Energy

States of low energy have been introduced by Olbermann and are quasifree (Gaussian) homogeneous isotropic pure state whose energy density ΔE_g smeared in time with a test function g is minimal.

$$\Delta E_g := \int dt\, g(t)^2(\omega(:\rho(t):))$$

On de Sitter spacetime we consider a state of low energy as the generalised ground state and construct a coherent state w.r.t. them.

We consider the massive case and for simplicity the case $m^2 = 2H^2$ (Degner).

The Klein-Gordon equation and Einstein tensor

The Klein-Gordon equation on de Sitter and with minimal coupling is:

$$\left(\frac{\partial^2}{\partial t^2} + 3H\frac{\partial}{\partial t} + m^2\right) f = 0$$

The zero-zero component of Einstein tensor in de Sitter spacetime is:

$$G_{00} = 3H^2$$

where H is the Hubble parameter.

Because the function f is a real solution of the Klein-Gordon equation, we conclude that $\rho(f(t)) > 0$ for all t. But $\lim_{t\to-\infty}\omega_{SLE}(:\rho:) = +\infty$ (Degner) and thus we can conclude that in the de Sitter spacetime there exists no solution for the semi-classical Einstein equation with coherent states w.r.t states of low energy for all t.

The semi-classical Einstein equation in Robertson-Walker spacetime is:

$$\omega_{SLE}(:\rho:) + \rho(f) = \frac{3H^2}{8\pi G} \equiv 3m_P^2 H^2 \tag{3}$$

where m_P is the *Planck mass* and we put $m_P = 1$.

The test function g, which is used to define the state of low energy in this spacetime, is a Gaussian:

$$g(t) = e^{-\frac{(t-t_0)^2}{\epsilon^2}}$$

If $\epsilon \gg \frac{1}{m_p}$, then the expectation value of the energy density evaluated in the state of low energy is much smaller than the right hand side of the relation (3), $\omega_{SLE}(:\rho:) \ll 3m_P^2 H^2$ (Degner, Dappiaggi/Hack/Pinamonti).

Therefore we neglect it and try to find a homogeneous and isotropic solution f of the Klein-Gordon equation which fulfils the semi-classical Einstein equation alone.

The Klein-Gordon Equation

The homogeneous and isotropic Klein-Gordon equation for conformal coupling constant is:

$$\left(\partial_t^2 + 3H\partial_t + m^2 + \frac{R}{6}\right) f(t) = 0 \tag{4}$$

where R is the Ricci scalar, in Robertson-Walker spacetime $R = 6(\dot{H} + 2H^2)$.

The Energy Density

The energy density with conformal coupling constant is given by:

$$\rho(f(t)) = \tfrac{1}{2}\dot{f}^2(t) + \tfrac{1}{2}(m^2 - H^2)f^2(t) \tag{5}$$

We define $h(t) := a(t)f(t)$ and change the time variable to $a(t)$, the scale factor, because do not know H explicitly as a function of t in the general case. Then the semi-classical Einstein equation and the Klein-Gordon equation become:

$$\frac{1}{2}H^2(\partial_a h)^2 - \frac{H^2}{a}(\partial_a h)h + \frac{m^2}{2a^2}h^2 = 3H^2$$

$$[(a^2 H\partial_a)^2 + m^2 a^2]h = 0$$

For the massless case the solution of Klein-Gordon equation $h(a(t))$ is given by:

$$h(a(t)) = \sqrt{6}a \sinh\left(c_1 \pm \log(a(t))\right)$$

and the Hubble parameter is given by:

$$H(a(t)) = \frac{c_2}{a^3}$$

This resulting Hubble parameter is different from the observed one, which is (with $H(a)|_{a=1} = \sqrt{3}$):

$$H^2 = \frac{\Omega_r}{a^4} + \frac{\Omega_m}{a^3} + \Omega_\Lambda$$

therefore we add the cosmological constant to the semi-classical Einstein equation and calculate the Hubble parameter for the massless case again.

The semi-classical Einstein equation with cosmological constant for the massless case is:

$$\frac{1}{2}H^2(\partial_a h)^2 - \frac{H^2}{a}(\partial_a h)h + \Omega_\Lambda = 3H^2$$

We solve the equation for the case $a \gg 1$, hence, the Hubble parameter is in order of $\sqrt{\Omega_\Lambda}$. Then the Hubble parameter turns out to be:

$$H^2(a(t)) = \frac{3}{4}\frac{c^2}{a^4} + \Omega_\Lambda + O(a^{-8})$$

- The semi-classical Einstein equation has been solved properly with a suitable coherent state for both massless and massive scalar free fields in the 3D torus spacetime.
- There exists no coherent state with respect to a state of low energy in de Sitter spacetime which can solve the semi-classical Einstein equation.
- We solved the semi-classical Einstein equation on general RW spacetimes with the classical energy density of the Klein-Gordon solution.

Thank you for your attention!