Spectral Action on Quantum Spheres

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INNOVATIVE ECONOMY
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DEVELOPMENT FUND

Motivation

- Why Spectral Action?
  - Idea: Do geometry using quantum tools - spectral approach
  - Geometric description of pathological spaces (fractals, non-Hausdorff spaces, foliations, ...)
  - "A dress for Standard Model - the beggar"
  - Testable predictions (cosmology, particle physics)
- Why Podleś (quantum) sphere?
  - A quantum homogeneous space of $SU_q(2)$
  - A $\mathcal{U}_q(su(2))$-equivariant spectral triple
  - First example of a "truly noncommutative" space $(\mathcal{A}_q, \mathcal{H}_q, \mathcal{D}_q)$
  - Peculiarities in the dimension spectrum

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$(\mathcal{A}, \mathcal{H}, \mathcal{D})$ - spectral triple

- $\mathcal{A}$ - pre-C^*-algebra (unital)
- $\mathcal{H}$ - Hilbert space
  $\exists$ a faithful representation $\pi(\mathcal{A}) \subset \mathcal{B}(\mathcal{H})$
- $\mathcal{D}$ - the Dirac operator - selfadjoint, unbounded
  - $(\mathcal{D} - \lambda)^{-1}$ for any $\lambda \notin \mathbb{R}$- compact resolvent
  - $[\mathcal{D}, \pi(a)] \in \mathcal{B}(\mathcal{H})$ for all $a \in \mathcal{A}$
  - $[[\mathcal{D}, \pi(a)], J\pi(b^*)J^\dagger] = 0$ for all $a, b \in \mathcal{A}$ - first order condition
- J - real structure, antilinear - $J^2 = \pm 1$, $J\mathcal{D} = \pm \mathcal{D}J$, $JaJ^{-1} \subset \pi(\mathcal{A})'$
- regularity axiom - ,,smoothness"

$$\pi(\mathcal{A}), [\mathcal{D}, \pi(\mathcal{A})] \subset \cap_{k=0}^{\infty} \mathrm{Dom}(\delta^k), \text{ with } \delta = [|\mathcal{D}|, \cdot]$$

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ΨDO calculus

- Unbounded derivation $\delta(T) := [|\mathcal{D}|, T]$ for any $T \in \mathcal{B}(\mathcal{H})$.
- $\mathrm{OP}^0 = \cap_{k\geq 0} \mathrm{Dom}\, \delta^k$ - operators of order ≤ 0.
- $\mathrm{OP}^\alpha = \{T \mid T|\mathcal{D}|^{-\alpha} \in \mathrm{OP}^0\}$.
- $D(\mathcal{A})$ polynomial algebra generated by $\mathcal{A}, J\mathcal{A}J^\dagger, \mathcal{D}, |\mathcal{D}|$.
- Pseudodifferential operators $\Psi(\mathcal{A}) \ni T \quad \Rightarrow$

$$T = P|\mathcal{D}|^{-p} \mod \mathrm{OP}^{-N}, \quad \text{for some } P \in D(\mathcal{A}),\ p \in \mathbb{N} \text{ and any } N \in \mathbb{N}.$$

- ΨDOs of order $\leq k$: $\Psi(\mathcal{A})^k := \Psi(\mathcal{A}) \cap \mathrm{OP}^k$.
- Examples: $a, \mathcal{D}^2, a[\mathcal{D}, b]|\mathcal{D}|^7, [\mathcal{D}, b]a|\mathcal{D}|^{-14}$.
- Regularity axiom $\Rightarrow \Psi(\mathcal{A})^\alpha \Psi(\mathcal{A})^\beta \subset \mathrm{OP}^{\alpha+\beta}$.

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- Regularity axiom $\Rightarrow \Psi(\mathcal{A})^\alpha \Psi(\mathcal{A})^\beta \subset \mathrm{OP}^{\alpha+\beta}$.

ΨDO calculus

- Unbounded derivation $\delta(T) := [|\mathcal{D}|, T]$ for any $T \in \mathcal{B}(\mathcal{H})$.
- $\mathrm{OP}^0 = \cap_{k \geq 0} \mathrm{Dom}\, \delta^k$ - operators of order ≤ 0.
- $\mathrm{OP}^\alpha = \{T \mid T|\mathcal{D}|^{-\alpha} \in \mathrm{OP}^0\}$.
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Dimension Spectrum - the definition

Dimension Spectrum [Connes, Moscovici (1995)]

A spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ has dimension spectrum Sd if $\mathrm{Sd} \subset \mathbb{C}$ is discrete and for any element b of the algebra $\Psi(\mathcal{A})^0$ the function

$$\zeta_{\mathcal{D}}^{b}(z) = \mathrm{Tr}\left(b|\mathcal{D}|^{-z}\right)$$

extends holomorphically to $\mathbb{C} \setminus \mathrm{Sd}$.

1. Higher order poles $\Rightarrow$ multiplicities in Sd.
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A commutative spectral triple

Connes' Reconstruction Theorem [1996-2008]

For every *commutative* spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ fulfilling the axioms there exists a smooth (compact) spin Riemannian manifold M such that:

- $\mathcal{A} = C^\infty(M)$,
- $\mathcal{H} = L^2(S(M))$,
- $\mathcal{D} = \not{D} = -\,\mathrm{i}\,\gamma^\mu \nabla^S_\mu$.

- $\mathrm{Sd}(M) = \dim(M) - \mathbb{N}$ and all of the poles are of first order.
- On manifolds with conical singularities one may encounter second order poles [J-M Lescure 1998].

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The Spectral Action Principle

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Physical action depends only upon the spectrum of $\mathcal{D}$.

Bosonic spectral action: $S_b = \operatorname{Tr} f\left(\mathcal{D}/\Lambda\right)$, Λ - energy scale and f - cut-off function

Theorem [Chamseddine, Connes (1997)]

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with $f_k = \int_0^\infty f(x) x^{k-1} dx$, $⨍ P = \underset{z=0}{\operatorname{Res}} \operatorname{Tr} P|\mathcal{D}|^{-z}$.

Main tools:

- Heat kernel expansion $\operatorname{Tr}\left(e^{-t\mathcal{D}^2}\right) \underset{t \downarrow 0}{\sim} \sum_{k \geq 0} a_k(\mathcal{D}^2) t^{(k-d)/2}$
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The algebra $\mathcal{A}_q$ is generated by $A = A^*, B, B^*$ fulfilling the relations

$$AB = q^2 BA, \qquad AB^* = q^{-2} B^* A, \qquad BB^* = q^{-2} A(1-A), \qquad B^* B = A(1 - q^2 A),$$

for some parameter $0 < q < 1$.

Theorem[Dabrowski, Sitarz (2003)]

There are two inequivalent $\mathcal{U}_q(su(2))$-equivariant reps $\pi_\pm$ of $\mathcal{A}_q$ on $\mathcal{H}_{\frac{1}{2}}$.

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The issue of regularity

- $(\mathcal{A}_q, \mathcal{H}, \mathcal{D})$ does not satisfy the regularity axiom!
  Although $\pi(\mathcal{A}_q), [\mathcal{D}, \pi(\mathcal{A}_q)] \in \mathcal{B}(\mathcal{H})$, but $[|\mathcal{D}|, [\mathcal{D}, \pi(\mathcal{A}_q)]] \notin \mathcal{B}(\mathcal{H})$.
- How to define ΨDO's? What should be the notion of dimension spectrum?
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$$|\mathcal{D}|[\mathcal{D}, \pi(a)] - \chi^{-1}[\mathcal{D}, \pi(a)]|\mathcal{D}| = B(z, a)|\mathcal{D}|^0, \qquad \text{with } \chi := \begin{pmatrix} q & 0 \\ 0 & q^{-1} \end{pmatrix}$$

Proposition [M.E., Iochum, Sitarz (2012)]

Let $\mathcal{B}$ be an algebra, generated by elements of the form

$$|\mathcal{D}|^z \omega |\mathcal{D}|^{-z}$$

for any $z \in \mathbb{C}$ and any $\omega \in \Omega_D(\mathcal{A}_q)$:

$$\omega = a_0[\mathcal{D}, a_1] \cdots [\mathcal{D}, a_k], \text{ where } k \geq 0 \text{ and } a_j \in \mathcal{A}_q, \text{ with } 0 \leq j \leq k.$$

Then $\mathcal{B} \subset \mathrm{op}^0(\mathcal{A}_q)$ *i.e.* it preserves $\mathrm{Dom}(|\mathcal{D}|^s) \subset \mathcal{H}$ for every $s \in \mathbb{R}$.

The dimension spectrum of the Podleś sphere

Theorem [M.E., Iochum, Sitarz (2012)]

For any $b \in \mathcal{B}$, the set of poles of the spectral function $\zeta_{\mathcal{D}}^{b} : s \mapsto \mathrm{Tr}(b|\mathcal{D}|^{-s})$ is a subset of $-\mathbb{N} + i\frac{2\pi}{\log q}\mathbb{Z}$ and the poles are at most of the second order.

Corollary

With the identification of $\mathcal{B}$ as the algebra of pseudodifferential operators of order ≤ 0, the dimension spectrum of the $\mathcal{U}_q(su(2))$-equivariant Podleś sphere is

$$\mathrm{Sd}(S_q^2) = -\mathbb{N} + i\frac{2\pi}{\log q}\mathbb{Z},$$

and the multiplicity of every point in $\mathrm{Sd}(S_q^2)$ is 2.

From dimension point of view S_q^2 behaves like a

- **point**, since it is 0-summable;
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"Pure gravity" spectral action

Proposition [M.E., Iochum, Sitarz (2012)]

For a suitable function f the spectral action on standard Podleś sphere reads

$$\operatorname{Tr} f\left(|\mathcal{D}|/\Lambda\right) = \sum_{\alpha \in \mathrm{Sd}_1} \sum_{n=0}^{2} a_{\alpha,n} \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} f_{\alpha,k} \left(\log \Lambda\right)^{n-k} \Lambda^{\alpha}$$

where $f_{\alpha,k} := \langle h_{\alpha,k}, f\rangle$ and $h_{\alpha,k} := \mathcal{L}^{-1}(s^{-\alpha} \log^k s)$ is a (distributional) inverse Laplace transform. The coefficients are determined by

$$a_{\alpha,n} := n! \operatorname*{Res}_{s=\alpha} \left((s-\alpha)^n \, \Gamma(s)\, \zeta_{\mathcal{D}}(s)\right)$$

$$\operatorname{Tr} f\left(|\mathcal{D}|/\Lambda\right) = c_2(|\mathcal{D}|, f) \log^2 \Lambda + c_1(|\mathcal{D}|, f) \log \Lambda + \\ + \sum_{a \in \mathbb{Z}^*} \left[d_1(|\mathcal{D}|, f, a) \log \Lambda + d_2(|\mathcal{D}|, f, a) \right] \Lambda^{ia} + O\left((\log^2 \Lambda)\, \Lambda^{-2} \right)$$

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Introducing gauge bosons

Fluctuation $\mathcal{D} \to \mathcal{D}_{\mathbb{A}} = \mathcal{D} + \mathbb{A}$, where $\mathbb{A} = \sum_i a_i[\mathcal{D}, b_i]$, for $a_i,\ b_i \in \mathcal{A}$, is selfadjoint.

$\mathbb{A}$ - a noncommutative one-form $\Leftrightarrow$ gauge potential

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Summary

- Podleś sphere - the first truly noncommutative space.
- From dimension point of view it behaves like a
  - **point**, since it is 0-summable;
  - **singular manifold**, since the poles are of second order;
  - **fractal** due to complex numbers in Sd.
- "q-regularity" instead of regularity
- An exact formula for the spectral action is available.
  - The leading term is $\log^2 \Lambda$.
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Thank you for your attention!

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Hopf algebra equivariance of spectral triples

H-equivariant module

Let V be an $\mathcal{A}$-module, H be a Hopf algebra and also let V and $\mathcal{A}$ be H-modules. We say that V is H-equivariant if

$$h(\alpha v) = \left(h_{(1)} \triangleright \alpha\right) \left(h_{(2)} v\right), \qquad \forall\, h \in H,\, \alpha \in \mathcal{A},\, v \in V$$

H-equivariant representation

A bounded representation π of $\mathcal{A}$ on $\mathcal{H}$ is called H-equivariant if there exists a dense linear subspace V of $\mathcal{H}$ such that V is an H-equivariant $\mathcal{A}$-module and $\pi(\alpha)v = \alpha v$, $\forall\, v \in V$, $\alpha \in \mathcal{A}$.

H-equivariant Dirac operator

$\mathcal{D}$ is H-equivariant if $\mathcal{D}h = h\mathcal{D}$, $\forall\, h \in H$.

$\mathcal{U}_q(su(2))$-equivariant rep of the standard Podleś sphere

$$
\begin{aligned}
A|l,m\rangle &= A^+_{l,m}|l+1,m\rangle + A^0_{l,m}|l,m\rangle + A^-_{l,m}|l-1,m\rangle,\\
B|l,m\rangle &= B^+_{l,m}|l+1,m+1\rangle + B^0_{l,m}|l,m+1\rangle + B^-_{l,m}|l-1,m+1\rangle,\\
B^*|l,m\rangle &= \widetilde{B}^+_{l,m}|l+1,m-1\rangle + \widetilde{B}^0_{l,m}|l,m-1\rangle + \widetilde{B}^-_{l,m}|l-1,m-1\rangle,
\end{aligned}
$$

$$
\begin{aligned}
B^+_{l,m} &:= q^m\sqrt{[l+m+1][l+m+2]}\,\alpha^+_l, & \widetilde{B}^+_{l,m} &:= q^{m-1}\sqrt{[l-m+2][l-m+1]}\,\alpha^-_{l+1},\\
B^0_{l,m} &:= q^m\sqrt{[l+m+1][l-m]}\,\alpha^0_l, & \widetilde{B}^0_{l,m} &:= q^{m-1}\sqrt{[l+m][l-m+1]}\,\alpha^0_l,\\
B^-_{l,m} &:= q^m\sqrt{[l-m][l-m-1]}\,\alpha^-_l, & \widetilde{B}^-_{l,m} &:= q^{m-1}\sqrt{[l+m][l+m-1]}\,\alpha^+_{l-1}
\end{aligned}
$$

$$
\begin{aligned}
A^+_{l,m} &:= -q^{m+l+\frac{1}{2}}\sqrt{[l-m+1][l+m+1]}\,\alpha^+_l\\
A^0_{l,m} &:= q^{-\frac{1}{2}}\tfrac{1}{1+q^2}\left([l-m+1][l+m]-q^2[l-m][l+m+1]\right)\alpha^0_l+\tfrac{1}{1+q^2},\\
A^-_{l,m} &:= q^{m-l-\frac{1}{2}}\sqrt{[l-m][l+m]}\,\alpha^-_l\ .
\end{aligned}
$$

Podleś sphere $\to S^2$

For $q \to 1$

- $\mathcal{A}(S^2_1) \cong C^\infty(S^2)$ with generators

$$A = \tfrac{1}{2}(1 - z), \qquad B = \tfrac{1}{2}(x + iy), \qquad x^2 + y^2 + z^2 = 1$$

- $H_{\frac{1}{2}} \cong L^2(S^2)$ with $|l, m\rangle_\pm = Y_l^m$
- $[x]_q \to x$, hence $\mathcal{D}_{S^2_q} \to \mathcal{D}_{S^2}$
- We have a q-analogue of the Hopf fibration

$$SU(2) \cong S^3 \xrightarrow{U(1)} S^2 \qquad\qquad SU_q(2) \cong S^3_q \xrightarrow{U(1)} S^2_q$$

Formulas for the Laplace transform

Lemma 1

For $k \in \mathbb{N}$ and $\alpha \in \mathbb{C} \setminus -\mathbb{N}$, the inverse Laplace transform $h_{\alpha,k}$ of the function $s \in \mathbb{R}^+ \mapsto s^{-\alpha} \log^k s$ reads

$$h_{\alpha,k}(t) = (-1)^k \frac{d^k}{d\alpha^k} \left(\frac{t^{\alpha-1}}{\Gamma(\alpha)} \right), \quad t > 0.$$

Lemma 2

The distributional inverse Laplace transform of $s \in \mathbb{R}^+ \mapsto \Theta(s) \log^k s$ where $0 \neq k \in \mathbb{N}$ is given by

$$\mathcal{L}^{-1}(\Theta(s) \log^k s)(t) = (-1)^k \Big[\sum_{j=1}^{k} \binom{k}{j} \gamma^{j-1} \left[\mathrm{Fp} \left(\frac{\Theta(t)}{t} \right) \right]^{*j} + \gamma^k \, \delta(t) \Big], \quad t > 0.$$

Lemma 3

The distributional inverse Laplace transform $h_{-n,k}$ of $s \in \mathbb{R} \mapsto \Theta(s)\, s^n \log^k s$ where $n \in \mathbb{N}$ and $0 \neq k \in \mathbb{N}$ is given by

$$h_{-n,k}(t) = (-1)^{k+n} n! \sum_{j=1}^{k} \binom{k}{j} \gamma^{j-1} \sum_{i=1}^{j} (-1)^{i+1} H(n)^{i-1} \, \mathrm{Fp} \left(\frac{\Theta(t)}{t^{n+1}} \right) * [\mathrm{Fp} \left(\frac{\Theta(t)}{t} \right)]^{*(j-i)}$$
$$+ (-1)^k [\gamma^{-1} \left((1 - \gamma\, H(n))^k - 1 \right) + \gamma^k] \, \delta^{(n)}(t), \quad t > 0.$$

The twist

- To find a proper Hochschild cocycle on S_q^2 one need to introduce a twist

$$\operatorname*{Res}_{z=0} \operatorname{Tr} \mathcal{D}^{-2z} \to \operatorname*{Res}_{z=2} \operatorname{Tr} K^{-2} \mathcal{D}^{-2z},$$

  with K - the group-like generator of $\mathcal{U}_q(\mathfrak{su}(2))$.
- The twist affects the dimension spectrum
  - S_q^2 becomes $2-$summable;
  - Sd becomes simple;
  - but the complex poles persist.
- The regularity axiom is still not satisfied.
- Twisted dimension spectrum, twisted spectral action?